ELECTRICAL ENGINEERING TEXTS

PRINCIPLES
OF
ALTERNATING CURRENTS

BY
RALPH R. LAWRENCE
Professor of Electrical Machinery at The Massachusetts Institute of Technology, Fellow of The American Institute of Electrical Engineers

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PREFACE TO THE SECOND EDITION

The use of the book by the author in class and suggestions from other teachers have indicated changes and amplifications in the text which were desirable. These changes have been made. New material on even harmonics, coupled circuits and a brief chapter on Electric Wave Filters have been added.

RALPH R. LAWRENCE.

CAMBRIDGE, MASSACHUSETTS,
January, 1935

PREFACE TO THE FIRST EDITION

This book has been developed from notes on Alternating Currents used for several years at the Massachusetts Institute of Technology with the junior students in Electrical Engineering. The portions of the notes dealing with single-phase currents were originally written by Professor H. E. Clifford, Gordon McKay Professor of Electrical Engineering at Harvard University, who was formerly Professor of Electrical Engineering at the Massachusetts Institute of Technology. The general arrangement and much of the material of these portions of the book are substantially in the same form as originally written.

No attempt has been made to include problems other than those used to illustrate the principles discussed, as two sets of problems on alternating currents, much more comprehensive than could have been incorporated in the book, were already published by Professor W. V. Lyon under the titles "Problems in Electrical Engineering"* and "Problems in Alternating Current Machinery."

* Now published under the title of "Problems in Alternating Currents."

The author wishes to express his indebtedness to Professor H. E. Clifford for the care with which he edited the manuscript and read the proof. The author also wishes to thank Professor W. V. Lyon for his many suggestions.

RALPH R. LAWRENCE.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE,
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NOTATION

In general the notation recommended by the American Institute of Electrical Engineers has been used. When the significance of the letters differs from that given in the following table, it is so stated in the text. A line over a letter indicates that it is a vector or complex quantity.

b = susceptance.
$\mathcal{B}$ = flux density.
C = capacitance.
D = distance.
d = distance.
E = root-mean-square value of a voltage or an electromotive force.
E_m = maximum value of a voltage or an electromotive force.
e = instantaneous value of a voltage or an electromotive force.
f = frequency in cycles per second.
f = function.
g = conductance.
h = height above the earth of the conductor of a transmission line.
$\mathcal{H}$ = field intensity.
I = root-mean-square value of a current.
I_m = maximum value of a current.
i = instantaneous value of a current.
j = operator which produces a counter-clockwise rotation of 90 degrees.
k = constant.
L = coefficient of self-induction, or self-inductance, or a length.
M = coefficient of mutual induction, or mutual inductance.
m = integer.
N = number of turns.
n = integer.
O = neutral point.
P = average power.
p = instantaneous power.
$p.f.$ = power-factor.
Q = root-mean-square value of a charge or steady value of a charge.
Q_m = Maximum value of a charge.
q = instantaneous value of a charge.
$\mathcal{R}$ = magnetic reluctance.
r = resistance.
S = coefficient of leakage induction, or leakage inductance.
$T = \frac{1}{f}$ = time in seconds of a complete cycle.

t = time in seconds.
V = root-mean-square value of a voltage or an electromotive force.
V_m = maximum value of a voltage or an electromotive force.
v = instantaneous value of a voltage or an electromotive force.
W = energy.
x = reactance.
y = admittance.
Z = number of inductors.
z = impedance.
α = phase angle and attenuation constant of a filter.
β = phase angle and phase constant of a filter.
γ = propagation constant of a filter.
ϵ = 2.718 = base of Napierian logarithms.
θ = phase angle.
λ = wave length.
μ = permeability.
$\omega = 2\pi f$ = angular velocity.
Σ = summation.
φ = flux.

PRINCIPLES OF ALTERNATING CURRENTS

CHAPTER I

ALGEBRA OF VECTORS AND OF COMPLEX QUANTITIES USED IN ELECTRICAL ENGINEERING

Quantities Involved in the Solution of Problems in Alternating Currents.—All quantities involved in the solution of ordinary problems in direct currents are simple algebraic quantities. All equations are simple algebraic equations and may be handled by any of the ordinary algebraic methods. These statements are equally true when applied to the currents, voltages, power etc., existing at any instant of time in an alternating-current circuit, *i.e.*, when applied to the so-called *instantaneous* values of current, voltage and power. Except in special cases, instantaneous values are not important. What is desired is the average power and the effective voltage and the effective current. Effective voltage and effective current cannot be handled by ordinary algebraic methods. They may be treated as vectors and must then be handled by methods which are applicable to vectors. For this reason, a knowledge of vector algebra and the algebra of complex quantities is necessary to an electrical engineer.

Types of Vectors Met in Electrical-engineering Problems.—There are two types of vectors, both of which occur in many alternating-current problems. There are vectors which lie in a plane and are fixed in direction, *i.e.*, *space vectors*, and vectors which are constant in magnitude and revolve in a plane with constant angular velocity, *i.e.*, *revolving vectors*. The latter are sometimes called *time vectors*. A constant force is a good example of a space vector. It acts in a fixed direction with a constant magnitude and may be represented in both direction and magnitude by a straight line whose length and direction represent

respectively, the magnitude and the direction of the force. An alternating current, which varies sinusoidally with time, may be represented by a revolving vector of fixed magnitude which revolves in a plane with constant angular velocity. If the length of the vector represents the maximum value of the current, its projection on a fixed reference axis is the value of the current at the instant of time considered. The number of revolutions per second made by the revolving line is equal to the number of cycles gone through by the current per second.

The phase difference between two vector quantities is the angle between the two vectors which represent the quantities.

Revolving vectors may be handled by any of the processes which are applicable to space vectors by merely considering them at some particular instant of time, or they may be treated by methods which are applicable to them alone.

Solution of Problems Involving Vectors by the Methods of Trigonometry.—Space vectors and revolving vectors which have equal angular velocities may be added or subtracted by the use of trigonometrical formulas for the solution of triangles, but when there are more than two vectors, this method of addition or subtraction becomes unnecessarily long and cumbersome and when applied to any but the simplest problems becomes hopelessly involved.

Vector Algebra.—The vector algebra which is necessary for handling problems in alternating currents is comparatively simple. It makes easy the solution of alternating-current problems which would otherwise be difficult. A knowledge of vector algebra is, therefore, one of the most useful tools to the electrical engineer.

There are two ways of treating vectors. They may be referred to rectangular coördinate axes and expressed in terms of their components along these axes or they may be expressed in terms of polar coördinates. Each of these methods has its advantages and both are useful. The former is better for addition and subtraction of vectors but can be used for multiplication and division of vectors as well. The latter is better when only multiplication and division are to be performed. It cannot be used for addition and subtraction. To change the expression of a vector from one form to the other is a simple matter.

Method of Complex Quantities.—The method of handling vectors when they are referred to coördinate axes is known as the *method of complex quantities.* In this method each vector is resolved into two components along and at right angles, respectively, to some conveniently chosen axis of reference. An operator j is attached to the component at right angles to the axis of reference to distinguish it from the component along that axis. The name *complex* as applied to this method does not indicate complexity of method. The method, so far as its application is concerned, should be called the *simplex method.* The name complex comes from the fact that each vector involved is resolved into so-called *real* and *imaginary* components, neither of which, however, is actually imaginary.

The components to which j is attached are called the imaginary components or simply *imaginaries,* or preferably the j components. The other components are called the real components or simply *reals.* The two rectangular axes along which these components lie are called the *axis of imaginaries* or the j axis and the axis of reals. The axis of reals is the axis from which the angles are measured which show the phase relations of the vectors. Although $j = \sqrt{-1}$ is an imaginary quantity, neither the component of the vector to which j is attached nor the axis along which it lies is imaginary. Both are real, just as real as the other component and the other axis.

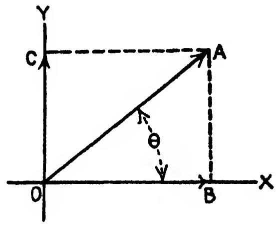


FIG. 1.

Let OA, Fig. 1, be a vector A making any angle θ with the reference axis OX.

Let the vector $OA = A$ be resolved into two components, $OB = B$ and $OC = C$, respectively along and at right angles to the axis OX. The vector has for its magnitude $\sqrt{B^2 + C^2}$ and makes an angle $\tan^{-1} \frac{C}{B}$ with OX. The expression for the vector A, in terms of its components, may be written

$$A = B + C \tag{1}$$

where the addition must be considered in a vector sense. To make it possible to distinguish between vector and non-vector

quantities in equations, short lines or dots over letters or numbers representing vector quantities may be used. Bold-face type is also used. For example,

$$\bar{A} = \bar{B} + \bar{C} \tag{2}$$

$$\dot{A} = \dot{B} + \dot{C} \tag{3}$$

$$\mathbf{A} = \mathbf{B} + \mathbf{C}$$

Sometimes the dot is placed under the vector instead of over it.

Since nearly all expressions met in alternating-current problems are vector expressions, the use of dots or dashes or bold-face type is often unnecessary. The simple algebraic expressions which occur are easily distinguishable from the vector expressions without the use of any special symbols.

Operator j.—Some notation must be adopted which will make it possible to distinguish readily between the components along the two axes. The letter j is used for this purpose. The letter j is an operator which indicates that a vector to which it is attached has been rotated through ninety degrees in a positive direction. Counter-clockwise direction is always considered positive and the horizontal direction is usually taken for the axis of reference. Left to right along this axis is considered positive.

The operator j does not differ, except in the effect it produces, from other common operators, such as plus and minus signs, multiplication and division signs, exponents and radical signs, log, sin, cos etc. For example, the exponent 3 with A^3 is merely an abbreviated way of writing $A \times A \times A$. The exponent 3 is an operator that indicates that a certain operation is to be performed on A, that is, it is to be multiplied by itself twice. In a similar way, the operator j indicates that the vector to which it is attached has been rotated through ninety degrees in a counter-clockwise direction. Using the operator j, equation (2) becomes

$$\bar{A} = B + jC \tag{4}$$

$B + jC$ is one form of vector expression for the vector $\bar{A}$. C, or in general the part of a vector to which j is attached, is the component at right angles to the reference axis, and B, or the component without j, is the component along the reference axis. C without the j attached would lie in a positive direction along

the reference axis. The letter j indicates that it has been rotated through ninety degrees in a positive direction from the axis of reference. Since the OX or horizontal axis was taken as the axis of reference, jC lies vertically upward or along the OY axis. The expression given for the vector $\bar{A}$ in equation (4) is known as its *complex expression*. The reason for the term *complex* will be explained later. It does not indicate complexity of expression. It has other significance.

Since the operator j rotates a vector to which it is attached through + 90 degrees, applying j twice or j^2 once rotates it through + 180 degrees. Applying j twice or j^2 once reverses a vector and is equivalent to multiplying the vector by -1. Therefore,

$$j \times j = j^2 = -1$$

and

$$j = \sqrt{-1}$$

Applying j three times or j^3 once gives

$$j^3 = j \times j^2 = j \times (-1) = -j$$

The operator $-j$, therefore, rotates any vector to which it is attached through -90 degrees or through 90 degrees in a clockwise or negative direction.

Representation of Vectors by the Use of the Operator j.—The four vectors A, jA, $j^2A = -A$ and $j^3A = -jA$ are shown in Fig. 2.

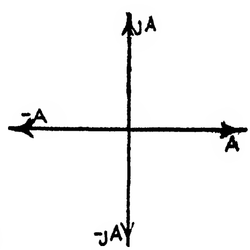


Fig. 2.

Four vectors of equal magnitude are given by the following equations:

$$\bar{A}_1 = a + jb \qquad (5)$$
$$\bar{A}_2 = -a + jb \qquad (6)$$
$$\bar{A}_3 = -a - jb \qquad (7)$$
$$\bar{A}_4 = a - jb \qquad (8)$$

These have magnitudes of $A = \sqrt{a^2 + b^2}$ and lie, respectively, in the first, second, third and fourth quadrants. They are shown in Fig. 3. Equations (5), (6), (7) and (8) are the complex expressions for the vectors.

The four vectors make angles, with the axis of reference, of α_1, α_2, α_3 and α_4, where

$$\tan \alpha_1 = \frac{b}{a} \qquad \sin \alpha_1 = \frac{b}{\sqrt{a^2 + b^2}} \qquad \cos \alpha_1 = \frac{a}{\sqrt{a^2 + b^2}} \qquad (9)$$

$$\tan \alpha_2 = \frac{b}{-a} \qquad \sin \alpha_2 = \frac{b}{\sqrt{a^2 + b^2}} \qquad \cos \alpha_2 = \frac{-a}{\sqrt{a^2 + b^2}} \qquad (10)$$

$$\tan \alpha_3 = \frac{-b}{-a} \qquad \sin \alpha_3 = \frac{-b}{\sqrt{a^2 + b^2}} \qquad \cos \alpha_3 = \frac{-a}{\sqrt{a^2 + b^2}} \qquad (11)$$

$$\tan \alpha_4 = \frac{-b}{a} \qquad \sin \alpha_4 = \frac{-b}{\sqrt{a^2 + b^2}} \qquad \cos \alpha_4 = \frac{a}{\sqrt{a^2 + b^2}} \qquad (12)$$

When expressing the value of the tangent of the angle α, which a vector makes with the reference axis OX, it should be left in the form of a ratio with the proper signs attached to both numerator and denominator. Unless this is done it is impossible

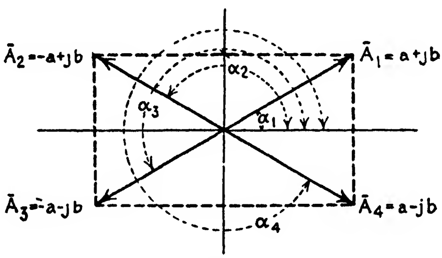


FIG. 3.

to tell in which of two quadrants the angle lies. Neither the sine nor the cosine of the phase angle is alone sufficient to fix the position of a vector. For example: referring to equations (9), (10), (11) and (12), $\sin \alpha = \dfrac{-b}{\sqrt{a^2 + b^2}}$ might refer to an angle in either the third or the fourth quadrant, and $\cos \alpha = \dfrac{-a}{\sqrt{a^2 + b^2}}$ might indicate an angle in either the second or the third quadrant.

Real and Imaginary or j Components of a Vector and Real and Imaginary or j Axes.—The two parts a and b of the vector expression $\bar{A} = a + jb$ are called, respectively, the *real* part and the *imaginary* or j part of the vector. The two axes along which they lie are called the *axis of reals* and the *axis of imaginaries* or the j **axis.** The expression $a + jb$ is called the *complex expression* of the vector $\bar{A}$. Neither the imaginary part b of the vector nor

the axis of imaginaries OY is imaginary. Both are just as real as the component a and just as real as the axis of reals. The term imaginary comes from the mathematical significance of the operator j which is attached to the so-called *imaginary component*. Mathematically, the operator $j = \sqrt{-1}$ is an imaginary quantity. When used as a rotating operator it does not make the component of a vector to which it is attached any less real than the so-called *real component*. For this reason, j component and j axis are better names than imaginary component and imaginary axis.

Operator ($\cos \alpha \pm j \sin \alpha$).—The four vectors A_1, A_2, A_3 and A_4, given in equations (5), (6), (7) and (8), page 5, and shown in Fig. 3, may be written

$$A_1 = A_1(\cos \alpha_1 + j \sin \alpha_1) \tag{13}$$
$$A_2 = A_2(\cos \alpha_2 + j \sin \alpha_2) \tag{14}$$
$$A_3 = A_3(\cos \alpha_3 + j \sin \alpha_3) \tag{15}$$
$$A_4 = A_4(\cos \alpha_4 + j \sin \alpha_4) \tag{16}$$

In all the equations the angles α are positive angles, *i.e.*, they are measured in a positive direction from the axis of reference.

If $\alpha_1 = 0$,

$$\bar{A}_1 = A_1(\cos 0 + j \sin 0) = A_1(1 + j0)$$

$\bar{A}_1$ lies, therefore, in a positive direction along the axis of reals. When α_1 has any value other than zero, such as β, the vector makes a positive angle β with the axis of reals. $(\cos \beta + j \sin \beta)$ is, therefore, an operator which rotates the vector A_1 through a positive angle β. In general,

$$(\cos \alpha + j \sin \alpha) \tag{17}$$

is an operator which rotates a vector to which it is applied through a positive angle α. It makes no difference whether the vector to which the operator is applied lies along the axis of reals or in any other direction, the operator rotates it from its original position through α degrees in a positive or counter-clockwise direction.

A little consideration will show that

$$(\cos \alpha - j \sin \alpha) \tag{18}$$

is an operator which rotates a vector to which it is applied through an angle α in a negative or clockwise direction from its original position.

In both of the operators, $(\cos \alpha + j \sin \alpha)$ and $(\cos \alpha - j \sin \alpha)$, the numerical value of cosine and sine must be taken for the positive angle α. Whether positive or negative rotation is produced depends on the sign attached to j in the operator. Although the sine and cosine are for the positive angle α, it must not be forgotten that the sine and cosine of certain angles are negative.

The general operator is $(\cos \alpha \pm j \sin \alpha)$.

If $\alpha = 0$,

$$\cos \alpha = 1 \text{ and } \sin \alpha = 0$$
$$\text{and } (\cos \alpha \pm j \sin \alpha) = 1$$

If $\alpha = 90°$,

$$\cos \alpha = 0 \text{ and } \sin \alpha = 1$$
$$\text{and } (\cos \alpha \pm j \sin \alpha) = \pm j$$

If $\alpha = 180°$,

$$\cos \alpha = -1 \text{ and } \sin \alpha = 0$$
$$\text{and } (\cos \alpha \pm j \sin \alpha) = -1$$

If $\alpha = 270°$,

$$\cos \alpha = 0 \text{ and } \sin \alpha = -1$$
$$\text{and } (\cos \alpha \pm j \sin \alpha) = \mp j$$

The single operator $(\cos \alpha + j \sin \alpha)$ may be used to rotate a vector in either a positive or a negative direction by giving the angle α the appropriate sign. If α is positive, the rotation produced is positive. If α is negative, the rotation produced is negative.

Operator which Rotates the References Axes through an Angle α.—In certain cases it is necessary to refer a vector to a new axis of reference which is displaced from the original axis by some definite angle, such as α. Rotating the axes with respect to a vector is equivalent to rotating the vector in the opposite direction with respect to the axes. Therefore, if the operator $(\cos \alpha + j \sin \alpha)$ rotates a vector through a positive angle α with respect to the axes, the same operator may be considered to rotate the axes through a negative angle α with respect to the vector.

The operators $(\cos\alpha - j\sin\alpha)$ and $(\cos\alpha + j\sin\alpha)$ are, therefore, two operators which applied to a vector rotate the axes with respect to the vector through an angle α in a positive and in a negative direction, respectively. As in the case of the operator $(\cos\alpha + j\sin\alpha)$ which rotates a vector to which it is applied in either a positive or a negative direction according to the sign of the angle α, the operator $(\cos\alpha - j\sin\alpha)$ may be made to serve for both positive and negative rotation of the axes by giving the angle α the appropriate sign. In problems there will frequently occur vectors which are referred to different axes. Before these vectors can be added or subtracted, multiplied or divided, they must all be referred to the same reference axis. This can easily be done, provided the angle between their reference axes is known, by applying the operator $(\cos\alpha \pm j\sin\alpha)$. The operator with the negative sign produces a positive or counter-clockwise rotation of the axes. With the positive sign the operator produces a negative or clockwise rotation of the axes.

Successive Application of Rotating Operators, Powers and Roots of Operators, Reciprocal of an Operator.—Consider the two operators

$$k_1 = \cos\theta_1 + j\sin\theta_1$$
$$k_2 = \cos\theta_2 + j\sin\theta_2$$

They rotate a vector to which they are applied through angles θ_1 and θ_2, respectively. Applied in succession they should rotate it successively through angles θ_1 and θ_2 or through a total angle $(\theta_1 + \theta_2)$.

$$\begin{aligned} k_1 \times k_2 &= (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2) \qquad (19) \end{aligned}$$

The product of two operators is thus a new operator which produces a rotation equal to the algebraic sum of the rotations produced by the operators individually. Similarly, the product of any number of rotating operators is a new operator which produces a rotation equal to the algebraic sum of the rotations produced by the operators separately.

$$
\begin{aligned}
k_1 \times k_2 \times k_3 \times \cdot\cdot \times k_n &= (\cos\theta_1 + j\sin\theta_1) \\
&\quad \times (\cos\theta_2 + j\sin\theta_2) \\
&\quad \times (\cos\theta_3 + j\sin\theta_3) \\
&\quad \times \cdot\cdot \times (\cos\theta_n + j\sin\theta_n) \\
&= \cos(\theta_1 + \theta_2 + \theta_3 + \cdot\cdot + \theta_n) \\
&\quad + j\sin(\theta_1 + \theta_2 + \theta_3 + \cdot\cdot + \theta_n) \\
&= \cos\Sigma\theta + j\sin\Sigma\theta \qquad (20)
\end{aligned}
$$

if θ_1, θ_2, θ_3 etc. are all equal,

$$k_1 \times k_2 \times k_3 \times \cdot\cdot \times k_n = k^n = \cos(n\theta) + j\sin(n\theta)$$

If $n\theta = 180$ degrees or π radians,

$$k^n = k^{\frac{\pi}{\theta}} = \cos\pi + j\sin\pi = -1 \qquad (21)$$

$$k = \sqrt[n]{-1} = j^{\frac{2}{n}} = \cos\frac{\pi}{n} + j\sin\frac{\pi}{n} \qquad (22)$$

Therefore, $\sqrt[n]{-1} = j^{\frac{2}{n}}$ is an operator which rotates a vector through $\frac{\pi}{n}$ radians. Since the operator $\sqrt[n]{-1}$ is equal to the nth root of minus one, it must have n roots or values, each of which produces a definite rotation. One of these roots is $\left(\cos\frac{\pi}{n} + j\sin\frac{\pi}{n}\right)$, which produces the rotation of $\frac{\pi}{n}$ radians.

Adding any number of $\pm 2\pi$ radians to an angle does not alter the value of its sine or of its cosine. Equation (21) may, therefore, be written

$$
\begin{aligned}
-1 &= \cos\pi + j\sin\pi \\
&= \cos(2q+1)\pi + j\sin(2q+1)\pi
\end{aligned}
$$

where q is any positive or negative integer.

$$j^{\frac{2}{n}} = \sqrt[n]{-1} = \cos\left(\frac{2q+1}{n}\right)\pi + j\sin\left(\frac{2q+1}{n}\right)\pi \qquad (23)$$

It might appear from equation (23) that the operator $j^{\frac{2}{n}}$ has an infinite number of values, since q may have an infinite number of values. There are, however, only n different roots of minus one. After the nth root, the roots repeat.

For example, let $n = 3$. In this case there are only three different roots. These are:

$$\text{For } q = 0, \quad j^{1/3} = \left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)$$

$$\text{For } q = 1, \quad j^{1/3} = \left(\cos\pi + j\sin\pi\right)$$

$$\text{For } q = 2, \quad j^{1/3} = \left(\cos\frac{5}{3}\pi + j\sin\frac{5}{3}\right)$$

For any greater values of q the roots repeat. For example, for $q = 3$, the root is $\frac{7}{3}\pi$, which is equivalent to $\frac{\pi}{3}$. This is the same as the first root. If q is given negative values, the same three roots result. For example, if $q = -1$, the root is $-\frac{\pi}{3}$ radians or -60 degrees. This is the same as $+300$ degrees or is the same as the third root obtained using positive values of q.

The operator $j = \sqrt{-1}$ has two roots, which produce, respectively, $+90$ and -90 degrees rotation. When using j, however, as an operator in the complex expression for a vector, the positive root is arbitrarily used. In such expressions j is used as an operator which produces a rotation of plus ninety degrees. Although the operator j is universally employed in work involving alternating currents to produce a rotation of $+90$ degrees, the roots of j which produce rotations of fractional parts of 90 degrees are not employed, as other more convenient forms of operator, which are single valued, are available for this purpose.

Reciprocal of the Operator (cos $\alpha \pm$ j sin α).—Consider the reciprocal of the operator $(\cos\alpha + j\sin\alpha)$, which produces a positive rotation of α degrees.

$$\begin{aligned}\frac{1}{\cos\alpha + j\sin\alpha} &= \frac{1}{\cos\alpha + j\sin\alpha} \times \frac{\cos\alpha - j\sin\alpha}{\cos\alpha - j\sin\alpha}\\ &= \frac{\cos\alpha - j\sin\alpha}{\cos^2\alpha + \sin^2\alpha} = \cos\alpha - j\sin\alpha \qquad (24)\end{aligned}$$

Therefore, the reciprocal of an operator which produces a rotation of α is an operator which produces a rotation of $-\alpha$. Dividing a vector by an operator which produces a rotation of

any angle α gives the same result as multiplying it by an operator which produces a rotation of α in the opposite direction. In general,

$$\frac{(\cos\alpha_1 + j\sin\alpha_1) \times (\cos\alpha_2 + j\sin\alpha_2) \times \cdots\cdots \times (\cos\alpha_n + j\sin\alpha_n)}{(\cos\beta_1 + j\sin\beta_1) \times (\cos\beta_2 + j\sin\beta_2) \times \cdots\cdots \times (\cos\beta_n + j\sin\beta_n)}$$
$$= \cos(\alpha_0 - \beta_0) + j\sin(\alpha_0 - \beta_0)$$

where α_0 is the algebraic sum of the angles α, and β_0 is the algebraic sum of the angles β.

Putting α in equation (24) equal to $\frac{\pi}{2}$ radians gives

$$\frac{1}{\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$
$$\frac{1}{j} = j^{-1} = -j$$

Therefore, the reciprocal of j or the minus-first power of j produces a rotation of -90 degrees.

Complex Operator Which Produces Uniform Angular Rotation. Let $\bar{A}$ be any vector of constant magnitude which rotates at a uniform angular velocity of $2\pi f = \omega$ radians per second. The number of revolutions made by the vector per second is f. Let time t be reckoned in seconds, and let it be considered zero when the vector $\bar{A}$ lies along the axis of reference in a positive direction. At any instant of time t seconds after $t = 0$, the vector will have rotated through ωt radians and will make an angle of ωt radians with the axis of reference, *i.e.*, the axis of reals. The operator which will produce this rotation is $(\cos\omega t + j\sin\omega t)$. By making use of this operator the vector $\bar{A}$ may be represented by

$$\bar{A} = A(\cos\omega t + j\sin\omega t) \tag{25}$$

The operator $(\cos\omega t + j\sin\omega t)$ produces a uniform rotation of $2\pi f = \omega$ radians per second.

$$\bar{A}' = A'[\cos(\omega t + \theta) + j\sin(\omega t + \theta)] \tag{26}$$

is another rotating vector that rotates with the angular velocity ωt and makes an angle θ radians with the time axis when $t = 0$. It leads (assuming θ positive) the vector $\bar{A}$ by θ radians. The angle θ is its phase angle, and, since the phase angle for $\bar{A}$ is zero, θ is the difference in phase between A' and A. In general, the phase difference between two vectors is the algebraic difference between their individual phase angles with respect to a common reference axis.

Solution of Vector Equations when the Vectors and Complex Quantities Involved Are Expressed in the Complex Form, i.e., in the Form a + jb.—In any vector equation the algebraic sum of the real terms on one side of the equation must be equal to the algebraic sum of the real terms on the other side of the equation, and similarly the algebraic sum of the imaginary or j terms on one side of the equation must be equal to the algebraic sum of the imaginary or j terms on the other side of the equation. Since the magnitude of a vector is equal to the square root of the sum of the squares of the real and the imaginary parts, the square of the sum of the real terms plus the square of the sum of the imaginary terms on one side of a vector equation must be equal to the square of the sum of the real terms plus the square of the sum of the imaginary terms on the other side of the equation. *The operator j is omitted in taking the squares as it is not a part of the magnitude of any component to which it is attached.*

In a direct-current circuit, current is given by voltage divided by resistance. A similar expression holds for the current in an alternating-current circuit. In the alternating-current circuit, the resistance must be replaced by the impedance. If the circuit is inductive, the numerical value of the impedance is $\sqrt{r^2 + x^2}$, where r is the resistance of the circuit and x is the reactance and is equal to the inductance of the circuit multiplied by 2π times the frequency of the impressed voltage.

The current multiplied by the resistance is the voltage drop in the circuit caused by the resistance. It is a vector which is in phase with the current. The current multiplied by the reactance is a voltage drop in the circuit caused by the reactance. It is a vector which is in quadrature with the current and leading it. Impedance is a complex quantity and its complex expression is $r + jx$.

$$\bar{V} = \bar{I}(r + jx)$$

from which we have

$$\bar{I} = \frac{\bar{V}}{r + jx}$$

Let $\bar{V} = 100 + j50$ and let the impedance be $4 + j3$. Then the magnitude of I is

$$I = \frac{\sqrt{(100)^2 + (50)^2}}{\sqrt{(4)^2 + (3)^2}} = \frac{111.8}{5} = 22.36 \text{ amperes}$$

The vector expression for the current is

$$\bar{I} = \frac{100 + j50}{4 + j3}$$

This must be rationalized, to get rid of the j in the denominator, by multiplying both numerator and denominator by the denominator with the sign of its j term reversed:

$$\begin{aligned} \bar{I} &= \frac{100 + j50}{4 + j3} \times \frac{4 - j3}{4 - j3} \\ &= \frac{400 - j300 + j200 + 150}{16 - j12 + j12 + 9} \\ &= \frac{550}{25} - j\frac{100}{25} = 22 - j4 \end{aligned}$$

The component of the current $\bar{I}$ which is along the axis to which $\bar{V}$ is referred is 22 and the component at right angles to the axis is 4. The magnitude of I is $\sqrt{(22)^2 + (4)^2} = 22.36$ amperes. The vector $\bar{I}$ makes an angle $\tan^{-1} \frac{-4}{22}$ with the reference axis.

$$\theta_V = \tan^{-1} \frac{50}{100} \qquad \theta_V = +26.6 \text{ degrees}$$

The vector diagram for $\bar{I}$ and $\bar{V}$ is shown in Fig. 4.

$$\theta_I = \tan^{-1} \frac{-4}{22} \qquad \theta_I = -10.3 \text{ degrees}$$

$$\theta_I^V = \theta_V - \theta_I = 26.6 - (-10.3) = 36.9 \text{ degrees}$$

It frequently happens, in solving a vector equation, that the magnitude and not the phase angle of one of the terms is known.

In such a case the equation cannot be solved as a vector equation. If, however, the term which cannot be expressed in vector form can be separated from the others, the equation may be solved by turning it into a simple algebraic equation by equating the square of the sum of the real terms plus the square of the sum of the imaginary or j terms on one side of the equation to the square of the term which cannot be expressed in vector form, which is on the other side. Consider the following example in which V is taken as the axis of reference, *i.e.*, as the axis of reals:

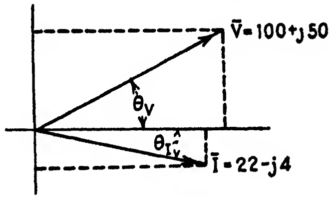


FIG. 4.

$$6000(\cos\theta + j\sin\theta) = V(\cos 0^\circ + j\sin 0^\circ) + 50(\cos 30^\circ + j\sin 30^\circ)\,(3 + j4) \qquad (27)$$

The angle θ is unknown. Substituting the values for the sines and cosines of the known angles in equation (27) gives

$$\begin{aligned} 6000(\cos\theta + j\sin\theta) &= V(1 + j0) \\ &\quad + 50(0.866 + j0.500)(3 + j4) \\ &= (V + 29.9) + j(248.2 + 0) \qquad (28) \\ (6000)^2 &= (V + 29.9)^2 + (248.2)^2 \\ V + 29.9 &= \sqrt{35{,}938{,}397} = \pm 5995 \\ V &= +5965 \quad \text{or} \quad -6025 \end{aligned}$$

Putting the positive value of V in equation (28) gives

$$\begin{aligned} \cos\theta + j\sin\theta &= \frac{5965 + 29.9}{6000} + j\frac{248.2}{6000} \\ &= 0.9992 + j0.0414 \\ \cos\theta &= 0.9992 \text{ and } \sin\theta = 0.0414 \\ \theta &= +2.37 \text{ degrees} \end{aligned}$$

Using the negative value of V gives

$$\begin{aligned} \cos\theta + j\sin\theta &= -0.9992 + j0.0414 \\ \cos\theta &= -0.9991 \text{ and } \sin\theta = 0.0414 \\ \theta &= +177.6 \text{ degrees} \end{aligned}$$

A vector diagram for equation (27) is shown in Fig. 5, in which the vectors corresponding to the negative solution are shown

dotted. The angles and vectors in Fig. 5 are not drawn exactly to scale. Their relative magnitudes are purposely altered to make the diagram clearer.

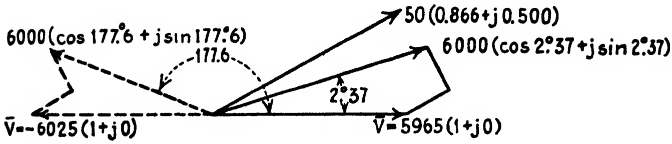


Fig. 5.

Exponential Operator $\epsilon^{\pm j\theta}$.—When vectors are to be added or subtracted, the operator ($\cos\alpha \pm j\sin\alpha$) should be used with each in order that the vectors may have the proper phase relations. Vectors when expressed in the complex form, *i.e.*, in the form $a + jb$, may be added or subtracted, multiplied or divided. When, however, the operations of multiplication and division only are to be performed, the exponential operator $\epsilon^{\pm j\theta}$ is much more convenient and should ordinarily be used. The exponential operator is particularly convenient in such cases since the processes of multiplication and division then involve only the addition and subtraction of exponents. The exponential form of operator should not be used when vectors are to be added or subtracted, since these operations cannot be performed by the mere addition and subtraction of exponents.

The expansion of $\epsilon^{j\theta}$ by Maclaurin's theorem[1] gives

$$\begin{aligned}\epsilon^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{\underline{|2}} + \frac{(j\theta)^3}{\underline{|3}} + \frac{(j\theta)^4}{\underline{|4}} + \frac{(j\theta)^5}{\underline{|5}} + \cdots \\ &= 1 + j\theta - \frac{\theta^2}{\underline{|2}} - j\frac{\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} + j\frac{\theta^5}{\underline{|5}} - \cdots \\ &= 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \cdots \\ &\qquad + j\left(\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \cdots\right) \qquad (29)\end{aligned}$$

where, for example, $\underline{|5}$ means $1 \times 2 \times 3 \times 4 \times 5$, and ϵ is the base of the Napierian logarithms, *i.e.*, 2.718.

[1] See any standard book on calculus.

Similarly, the expansion of the sine and cosine of θ gives

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \cdots \tag{30}$$

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \cdots \tag{31}$$

Therefore,

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta \tag{32}$$

The expansion of $\epsilon^{-j\theta}$ gives

$$\epsilon^{-j\theta} = 1 + (-j\theta) + \frac{(-j\theta)^2}{\lfloor 2} + \frac{(-j\theta)^3}{\lfloor 3} + \frac{(-j\theta)^4}{\lfloor 4} + \frac{(-j\theta)^5}{\lfloor 5} + \cdots$$

$$= 1 - j\theta - \frac{\theta^2}{\lfloor 2} + j\frac{\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} - j\frac{\theta^5}{\lfloor 5} - \cdots$$

$$= 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \cdots$$

$$-j\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \cdots\right)$$

$$= \cos\theta - j\sin\theta \tag{33}$$

$\epsilon^{j\theta}$ and $\epsilon^{-j\theta}$ are, therefore, two operators producing on a vector to which they are applied the same effect as the operators $(\cos\theta + j\sin\theta)$ and $(\cos\theta - j\sin\theta)$. $\epsilon^{j\theta}$ is an operator which rotates a vector to which it is applied through an angle θ in a positive direction. $\epsilon^{-j\theta}$ is an operator which rotates a vector to which it is applied through an angle θ in a negative direction. The angle θ in the two operators should be expressed in radians to be mathematically correct, but it is usually more convenient to express it in degrees.

$$A(\cos\theta + j\sin\theta) = A\epsilon^{j\theta} \tag{34}$$

$$A(\cos\theta - j\sin\theta) = A\epsilon^{-j\theta} \tag{35}$$

represent two vectors which make angles of plus θ and minus θ with the axis of reals.

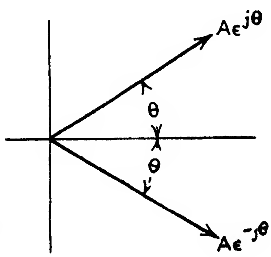


FIG. 6.

The operator $(\cos\theta \pm j\sin\theta)$ refers a vector to which it is applied to rectangular coördinates. The operator $\epsilon^{+j\theta}$ refers a vector to polar coördinates. The latter operator is called the

exponential operator. The vectors given in equations (34) and (35) are shown in Fig. 6.

In the above expressions, θ is the magnitude of the angle. The sign of the angle is taken care of by the sign before the j component in the operator. (See page 7.)

Let $\bar{x} = (a + jb)$ and $\bar{y} = (c + jd)$ be two vectors making angles $\theta_x = \tan^{-1}\frac{b}{a}$ and $\theta_y = \tan^{-1}\frac{d}{c}$ with some common reference axis. Then

$$\begin{aligned}\bar{x}\bar{y} &= (a + jb)(c + jd) \\ &= xy(\cos\theta_x + j\sin\theta_x)(\cos\theta_y + j\sin\theta_y) \\ &= xy\{\cos(\theta_x + \theta_y) + j\sin(\theta_x + \theta_y)\} \\ &= xy\epsilon^{j(\theta_x+\theta_y)} \qquad (36)\end{aligned}$$

Similarly, if $\bar{q} = q(\cos\theta_q + j\sin\theta_q)$ and $\bar{z} = z(\cos\theta_z - j\sin\theta_z)$ are two other vectors,

$$\begin{aligned}\bar{q}\bar{z} &= qz(\cos\theta_q + j\sin\theta_q)(\cos\theta_z - j\sin\theta_z) \\ &= qz\{\cos(\theta_q - \theta_z) + j\sin(\theta_q - \theta_z)\} \\ &= qz\epsilon^{j(\theta_q-\theta_z)} \qquad (37)\end{aligned}$$

$$\begin{aligned}\frac{\bar{x}}{\bar{y}} &= \frac{a + jb}{c + jd} = \left(\frac{x}{y}\right)\left(\frac{\cos\theta_x + j\sin\theta_x}{\cos\theta_y + j\sin\theta_y}\right) \\ &= \frac{x}{y}\left(\frac{\cos\theta_x + j\sin\theta_x}{\cos\theta_y + j\sin\theta_y}\right)\left(\frac{\cos\theta_y - j\sin\theta_y}{\cos\theta_y - j\sin\theta_y}\right) \\ &= \frac{x}{y}\left\{\frac{\cos(\theta_x - \theta_y) + j\sin(\theta_x - \theta_y)}{\cos^2\theta_y + \sin^2\theta_y}\right\} \\ &= \frac{x}{y}\{\cos(\theta_x - \theta_y) + j\sin(\theta_x - \theta_y)\} \\ &= \frac{x}{y}\epsilon^{j(\theta_x-\theta_y)} \qquad (38)\end{aligned}$$

An Example of the Use of the Exponential Operator $\epsilon^{\pm j\theta}$.—Suppose that the product of three vectors, of magnitudes 10, 15 and 20 making angles of +15, +20 and −120 degrees, respectively, with a fixed reference axis, is to be divided by a fourth vector of magnitude 10 making an angle of + 10 degrees with the same reference axis. The four vectors are

$$\begin{aligned}10\{\cos(+15°) + j\sin(+15°)\} &= 10\epsilon^{j15°} \\ 15\{\cos(+20°) + j\sin(+20°)\} &= 15\epsilon^{j20°} \\ 20\{\cos(-120°) + j\sin(-120°)\} &= 20\epsilon^{-j120°} \\ 10\{\cos(+10°) + j\sin(+10°)\} &= 10\epsilon^{j10°}\end{aligned}$$

To be mathematically correct the angles in the exponents of the ϵ's should be in radians. It is, however, more convenient to leave them in degrees.

The result of the operation of multiplication and division, using the exponential operators, is

$$\begin{aligned} \bar{A} &= \left(\frac{10 \times 15 \times 20}{10}\right)\left(\frac{\epsilon^{j15^\circ} \times \epsilon^{j20^\circ} \times \epsilon^{-j120^\circ}}{\epsilon^{j10^\circ}}\right) \\ &= 300\epsilon^{j(15^\circ+20^\circ-120^\circ-10^\circ)} \\ &= 300\,\epsilon^{-j95^\circ} \\ &= 300(\cos 95^\circ - j \sin 95^\circ) \\ &= 300(-0.0872 - j0.9962) \\ &= -\ 26.16 - j298.9 \end{aligned}$$

If the result had been obtained by using the operator $(\cos \theta \pm j \sin \theta)$ for rectangular coördinates, it would first have been necessary to rationalize the expression. There would then have been four quantities of the form $(a \pm jb)$ to multiply together in the numerator and two to multiply together in the denominator. To get the result in the final form of $(a \pm jb)$, the real terms in the numerator would then have to be added and divided by the denominator to get the real or a part of the resultant vector and the j terms in the numerator would have to be added and divided by the denominator to get the j or b part of the resultant vector. The saving of time by the use of the exponential operator is obvious in a case of this kind.

Exponential Operator which Produces Uniform Rotation.—If the angle θ in the operator $\epsilon^{\pm j\theta}$ is replaced by an angle which is proportional to time, the operator produces continuous rotation of any vector to which it is applied. Let $\theta = \omega t$, where ω is an angular velocity of $2\pi f$ radians per second and t is the time measured in seconds. Then $\epsilon^{\pm j\omega t}$ is an operator producing a continuous rotation of $2\pi f$ radians or f revolutions per second.

$$\bar{A} = A\epsilon^{+j\omega t} \tag{39}$$

$$\bar{A}' = A'\epsilon^{-j\omega t} \tag{40}$$

are two rotating vectors which have magnitudes A and A' and rotate with a uniform angular velocity of $2\pi f = \omega$ radians per second. They make f revolutions per second. $\bar{A}$ revolves in a

positive or counter-clockwise direction. $\bar{A}'$ revolves in a negative or clockwise direction.

$$\bar{B} = B\epsilon^{j\omega t} \tag{41}$$
$$\bar{C} = C\epsilon^{j(\omega t - \theta)} \tag{42}$$

are two rotating vectors having magnitudes of B and C and revolving in a counter-clockwise direction with an angular velocity of $\omega = 2\pi f$ radians per second. $\bar{C}$ lags behind $\bar{B}$ by θ radians. These continuously rotating vectors may be expressed in terms of rectangular coördinates instead of in polar coördinates by the use of the operator ($\cos \omega t \pm j \sin \omega t$). (See page 12.)

$$\bar{B} = B\epsilon^{j\omega t} = B\{\cos \omega t + j \sin \omega t\} \tag{43}$$
$$\bar{C} = C\epsilon^{j(\omega t - \theta)} = C\{\cos(\omega t - \theta) + j \sin(\omega t - \theta)\} \tag{44}$$

Polar Form of Operator.—It is often convenient to represent a vector by writing its magnitude followed by a phase angle which gives its phase position with respect to the reference axis. For example,

$$\bar{A} = A\underline{|\theta} \tag{45}$$
$$\bar{A}' = A\underline{|-\theta} \tag{46}$$

are two vectors of the same magnitude A, making angles of $+\theta$ and $-\theta$, respectively, with the reference axis. The two vectors $A\underline{|\theta}$ and $A\underline{|-\theta}$ are shown in Fig. 7.

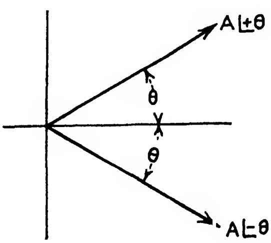


Fig. 7.

The symbolic operators

$$\underline{|+\theta} = \underline{|\theta} \text{ and } \underline{|-\theta} = \overline{|\theta}$$

indicate that the vectors to which they are attached have been rotated through angles of $+\theta$ and $-\theta$, respectively. Instead of using the sign with the angle to indicate whether it is positive or negative, the position of the bracket may be made to serve this purpose. When this is done, $\underline{|\theta}$ means a positive angle and $\overline{|\theta}$ means a negative angle. The polar form of operator is simpler to write and to print than the exponential form and for this reason is more often used.

Example of the Use of the Polar Form of Operator.—Suppose it is necessary to divide the product of three vectors, of magnitudes A, B and C, making angles of θ_A, $-\theta_B$ and θ_C, respectively, with some reference axis, by the product of two other vectors of magnitudes D and E making angles respectively of $-\theta_D$ and θ_E with the same axis. The vector representing the result of this operation is

$$\begin{aligned}
\frac{(\bar{A})(\bar{B})(\bar{C})}{(\bar{D})(\bar{E})} &= \frac{(A\,\underline{|+\theta_A})(B\,\underline{|-\theta_B})(C\,\underline{|+\theta_C})}{(D\,\underline{|-\theta_D})(E\,\underline{|+\theta_E})} \\
&= \frac{(A)(B)(C)}{(D)(E)}\,\underline{|(\theta_A-\theta_B+\theta_C)-(-\theta_D+\theta_E)} \\
&= \frac{(A)(B)(C)}{(D)(E)}\{\cos(\theta_A-\theta_B+\theta_C+\theta_D-\theta_E) \\
&\qquad + j\sin(\theta_A-\theta_B+\theta_C+\theta_D-\theta_E)\}
\end{aligned}$$

Example of the Solution of a Vector Equation Having Two Terms, Each Being Expressed by Means of the Polar Form of Operator.—The current in a certain transmission line under load conditions is given by the following vector equation in which the voltage at the receiving end is used as the axis of reference, *i.e.*, as the axis of reals.

$$\begin{aligned}
\bar{I} &= 175\,\underline{|-25^\circ.84} \times 0.9919\,\underline{|+0^\circ.313} \\
&\quad + 63{,}510\,\underline{|+0^\circ.0} \times 0.002543\,\underline{|+15^\circ.38} \times 0.1386\,\underline{|+74^\circ.72} \\
&= 175 \times 0.9919\,\underline{|-25^\circ.84 + 0.313} \\
&\quad + 63{,}510 \times 0.002543 \times 0.1386\,\underline{|+0^\circ.0 + 15^\circ.38 + 74^\circ.72} \\
&= 173.6\,\underline{|-25^\circ.53} + 22.38\,\underline{|+90^\circ.10} \\
&= 173.6\,(\cos 25^\circ.53 - j\sin 25^\circ.53) \\
&\qquad + 22.38\,(\cos 90^\circ.10 + j\sin 90^\circ.10) \\
&= 173.6\,(0.9024 - j0.4310) + 22.38\,(-0.0017 + j1.000) \\
&= 156.6 - j52.43 \\
I &= \sqrt{(156.6)^2 + (52.43)^2} \\
&= 165.2 \text{ amperes} \\
\tan\theta &= \frac{-52.43}{156.6} = -0.3347 \\
\theta &= -18^\circ.51 \\
\bar{I} &= 165.2\,\underline{|-18^\circ.51} = 165.2\,\overline{|18^\circ.51} \\
&= 165.2\epsilon^{-j18^\circ.51}
\end{aligned}$$

Square Root, Product and Ratio of Vectors or of Complex Quantities by Use of the Exponential Operator.—A method of getting the root or power of a vector or complex quantity by the use of the operator ($\cos\alpha \pm j\sin\alpha$) has been given on page 9. Roots and powers may be obtained much more easily by using the exponential operator. Let

$$\bar{A} = a + jb$$

be any vector making an angle α with the axis of reference. Then

$$\bar{A} = A\epsilon^{j\alpha}$$

A vector is not altered in direction or magnitude by multiplying it by an operator which rotates it through any whole number of 2π radians, *i.e.*, by multiplying it by the operator $\epsilon^{j2\pi q}$ where q is any integer. Therefore,

$$\bar{A} = A\epsilon^{j\alpha} = A\epsilon^{j\alpha}\,\epsilon^{j2\pi q} = A\epsilon^{j(\alpha+2\pi q)}$$

$$\sqrt{\bar{A}} = \sqrt{A}\,\epsilon^{j\left(\frac{\alpha}{2}+\pi q\right)} \tag{47}$$

There are only two different roots of the vector $\bar{A}$. There are two roots of $\sqrt{A}$, one plus and one minus. There are also two roots of the operator $\epsilon^{j\left(\frac{\alpha}{2}+\pi q\right)}$. These are for $q = 0$ and $q = 1$. For values of q greater than 1 the roots repeat.

For $q = 0$,

$$\sqrt{\epsilon^{j\alpha}} = \epsilon^{j\frac{\alpha}{2}}$$

For $q = 1$,

$$\sqrt{\epsilon^{j\alpha}} = \epsilon^{j\left(\frac{\alpha}{2}+\pi\right)}$$

For $q = 2$,

$$\sqrt{\epsilon^{j\alpha}} = \epsilon^{j\left(\frac{\alpha}{2}+2\pi\right)} = \epsilon^{j\frac{\alpha}{2}}$$

Adding π to the exponent of the operator is equivalent to reversing its sign. The two roots of the operator are, therefore,

$$\pm\epsilon^{j\frac{\alpha}{2}}$$

The two roots of the vector are

$$\sqrt{\bar{A}} = (\pm\sqrt{A})(\pm\,\epsilon^{j\frac{\alpha}{2}})$$
$$= \pm\sqrt{A}\epsilon^{j\frac{\alpha}{2}} = \pm\sqrt{A}\left(\cos\frac{\alpha}{2} + j\sin\frac{\alpha}{2}\right) \qquad (48)$$

The square root of a vector has, therefore, two values. It is a new vector having a magnitude equal to the square root of the magnitude of the original vector and a phase angle equal either to half the phase angle of the original vector or to π plus half the phase angle of the original vector.

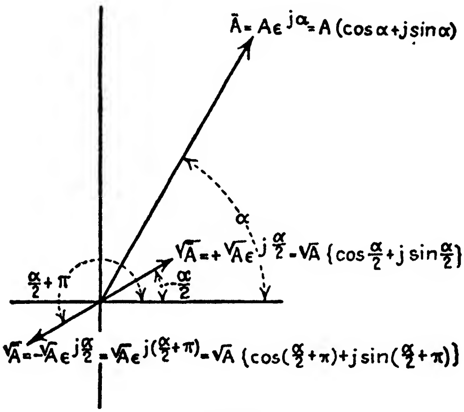


FIG. 8.

The two values of the square root of a vector are shown in Fig. 8.

Let $\bar{B} = B\epsilon^{j\beta}$ be a second vector of magnitude B making an angle β with the same reference axis that was used for the vector $\bar{A}$. Then

$$\bar{A}\bar{B} = A\epsilon^{j\alpha}B\epsilon^{j\beta}$$
$$= AB\epsilon^{j(\alpha+\beta)} = AB\{\cos(\alpha+\beta) + j\sin(\alpha+\beta)\} \qquad (49)$$

The product of two vectors is a new vector having a magnitude equal to the product of the magnitudes of the original vectors. It makes an angle with the reference axis equal to the sum of the phase angles of the original vectors. This was shown on page 9 by the use of the operator $(\cos\theta + j\sin\theta)$.

$$\frac{\bar{A}}{\bar{B}} = \frac{A\epsilon^{j\alpha}}{B\epsilon^{j\beta}} = \frac{A}{B}\epsilon^{j(\alpha-\beta)} = \frac{A}{B}\{\cos(\alpha-\beta) + j\sin(\alpha-\beta)\} \tag{50}$$

The ratio of two vectors is a new vector having a magnitude equal to the ratio of the magnitudes of the original vectors. It is displaced from the reference axis by an angle equal to the difference of the phase angles of the original vectors.

$$\frac{1}{\bar{A}} = \frac{1}{A\epsilon^{j\alpha}} = \frac{1}{A}\epsilon^{-j\alpha} = \frac{1}{A}(\cos\alpha - j\sin\alpha) \tag{51}$$

The reciprocal of a vector is a new vector having a magnitude equal to the reciprocal of the magnitude of the original vector. It is displaced from the reference axis by an angle equal in magnitude but opposite in direction to the phase angle of the original vector.

$$\begin{aligned}\sqrt{\bar{A}\bar{B}} &= \pm\sqrt{AB}\,\epsilon^{j\left(\frac{\alpha+\beta}{2}\right)}\\ &= \pm\sqrt{AB}\left\{\cos\left(\frac{\alpha+\beta}{2}\right) + j\sin\left(\frac{\alpha+\beta}{2}\right)\right\}\end{aligned} \tag{52}$$

$$\begin{aligned}\sqrt{\frac{\bar{A}}{\bar{B}}} &= \pm\sqrt{\frac{A}{B}}\,\epsilon^{j\left(\frac{\alpha-\beta}{2}\right)}\\ &= \pm\sqrt{\frac{A}{B}}\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + j\sin\left(\frac{\alpha-\beta}{2}\right)\right\}\end{aligned} \tag{53}$$

The fundamental constants which appear in the exact solution of the transmission line for current and voltage are the square root of the product and the square root of the ratio of the series impedance per unit length of conductor and the parallel admittance to neutral, also per unit length of conductor. Impedance and admittance are not vectors. They are complex quantities. They are really special operators. If a current vector is operated on with impedance, the result is a voltage vector which differs both in phase and in magnitude from the current vector. Similarly, if a voltage vector is operated on with admittance, the result is a current vector which differs both in magnitude and in phase from the voltage vector. So far as concerns the purely mechanical processes of multiplication, division, extraction of roots etc. of complex quantities, they are carried out in the same manner for complex quantities as for vectors. For the solution of the

equations for the transmission line, it is necessary to be able to take the square root of the product and the square root of the ratio of complex quantities.

Complex Quantity.—A complex quantity is a quantity which multiplied by a vector gives a new vector of a different magnitude from the original vector and displaced from it in phase. A complex quantity is a simple algebraic quantity combined with an operator. For example, the inductive impedance of an electric circuit is a complex quantity. It is written

$$r + jx = z\epsilon^{j\theta} = z(\cos\theta + j\sin\theta)$$

where the letters have the following significance:

r = resistance
$x = 2\pi fL$
f = frequency
L = inductance
$z = \sqrt{r^2 + x^2}$
$j = \sqrt{-1}$ = operator which rotates through +90 degrees
ϵ = base of the Napierian logarithms
$\theta = \tan^{-1}\dfrac{x}{r}$

Impedance multiplied by current gives voltage. This voltage is displaced from the current by an angle θ.

In general, the negative root obtained when taking the square root of a complex quantity, such as impedance, has no significance and does not have to be considered.

The nth Root and nth Power of Vectors and Complex Quantities.—In ordinary electrical-engineering problems, it is not often necessary to use roots or powers of vectors or complex quantities higher than the second. Higher roots and powers can be obtained, however, if necessary. A method for obtaining them using the operator $(\cos\theta + j\sin\theta)$ has already been given. They may be found more easily by using the operator $\epsilon^{j\theta}$. For example, if $\bar{A} = A\epsilon^{j\alpha}$,

$$\sqrt[n]{\bar{A}} = \sqrt[n]{A}\,\epsilon^{j\left(\frac{\alpha+2\pi q}{n}\right)} = \sqrt[n]{A}\left\{\cos\left(\frac{\alpha+2\pi q}{n}\right) + j\sin\left(\frac{\alpha+2\pi q}{n}\right)\right\} \quad (54)$$

where q is any integer.

This has n different roots.

$$\bar{A}^n = (A\epsilon^{j\alpha})^n = A^n\epsilon^{jn\alpha} = A^n(\cos n\alpha + j \sin n\alpha) \tag{55}$$

$$\begin{aligned}(\bar{A}\bar{B})^n = (A\epsilon^{j\alpha}B\epsilon^{j\beta})^n &= A^nB^n\epsilon^{jn(\alpha+\beta)} \\ &= A^nB^n\{\cos n(\alpha+\beta) \\ &\qquad + j \sin n(\alpha+\beta)\}\end{aligned} \tag{56}$$

$$\begin{aligned}\left(\frac{\bar{A}}{\bar{B}}\right)^n = \left(\frac{A\epsilon^{j\alpha}}{B\epsilon^{j\beta}}\right)^n &= \frac{A^n}{B^n}\epsilon^{jn(\alpha-\beta)} \\ &= \frac{A^n}{B^n}\{\cos n(\alpha-\beta) + j \sin n(\alpha-\beta)\}\end{aligned} \tag{57}$$

Logarithm of a Complex Quantity or a Vector.—Let $\bar{A} = a + jb$ be any complex quantity or vector. This may be written

$$\bar{A} = a + jb = A(\cos \alpha + j \sin \alpha) = A\epsilon^{j\alpha}$$

where $\alpha = \tan^{-1}\dfrac{b}{a}$.

$$\begin{aligned}\log_\epsilon \bar{A} &= \log_\epsilon A\epsilon^{j\alpha} \\ &= \log_\epsilon A + j\alpha \log_\epsilon \epsilon \\ &= \log_\epsilon A + j\alpha \\ &= \log_\epsilon \sqrt{a^2+b^2} + j\left(\tan^{-1}\frac{b}{a}\right)\end{aligned} \tag{58}$$

The logarithm of a vector or a complex quantity is a new vector or complex quantity having a real part equal to the logarithm of the magnitude of the original vector or complex quantity and an imaginary or j part equal to the phase angle, expressed in radians, of the original vector or complex quantity.

Representation of an Oscillating Vector, Whose Magnitude Varies Sinusoidally with Time, by the Use of Two Exponential Rotating Operators.—The motion of the end of a spiral spring that has been compressed and released is an example of an oscillating vector whose magnitude varies sinusoidally with time. A sinusoidal alternating current, *i.e.*, one whose magnitude varies sinusoidally with time, may be represented by a similar vector.

Any simple harmonic oscillating vector may be resolved into two oppositely rotating vectors, each of the same period as the given vector and of half its magnitude.

In Fig. 9, $\bar{A}$ is a vector, of maximum value A, which oscillates in magnitude between the limits of $+A$ and $-A$. The magnitude of this vector at each instant is given by

$$a = A \sin (\omega t + 90°) = A \cos \omega t$$

where ω is equal to 2π times the frequency of vibration of the vector $\bar{A}$. It is also equal to the angular velocity of the rotating vectors which replace $\bar{A}$. $\bar{A}_1$ and $\bar{A}_2$, each equal in magnitude to $\frac{A}{2}$, are the two oppositely rotating vectors which replace the vector $\bar{A}$. They are two conjugate rotating vectors. Their

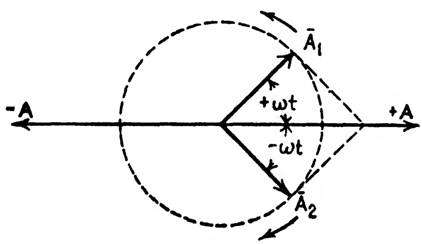


FIG. 9.

resultant always lies along the original vector and is equal to the magnitude of that vector at each instant.

Referring the two rotating vectors to $+A$ as a reference axis gives

$$\bar{A}_1 = \frac{A}{2} (\cos \omega t + j \sin \omega t) = \frac{A}{2}\epsilon^{+j\omega t}$$

$$\bar{A}_2 = \frac{A}{2} (\cos \omega t - j \sin \omega t) = \frac{A}{2}\epsilon^{-j\omega t}$$

$$a = A \cos \omega t = \bar{A}_1 + \bar{A}_2 = \frac{A\epsilon^{j\omega t} + A\epsilon^{-j\omega t}}{2} \tag{59}$$

$$a' = A \sin \omega t = \frac{\bar{A}_1 - \bar{A}_2}{j} = \frac{A\epsilon^{j\omega t} - A\epsilon^{-j\omega t}}{j2}$$

$$= \frac{A\epsilon^{+j\left(\omega t - \frac{\pi}{2}\right)} + A\epsilon^{-j\left(\omega t - \frac{\pi}{2}\right)}}{2} \tag{60}$$

If i_1 is the instantaneous value of a sinusoidal alternating current whose maximum value is I_{m1},

$$i_1 = I_{m1} \sin \omega t = \frac{I_{m1}\epsilon^{+j\left(\omega t - \frac{\pi}{2}\right)} + I_{m1}\epsilon^{-j\left(\omega t - \frac{\pi}{2}\right)}}{2} \tag{61}$$

In this expression i_1 is zero when time t is zero.

If i_2 is another sinusoidal alternating current of the same frequency as i_1, but lagging it in time phase by an angle θ,

$$i_2 = I_{m2} \sin (\omega t - \theta) = \frac{I_{m2}\epsilon^{+j\left(\omega t - \frac{\pi}{2} - \theta\right)} + I_{m2}\epsilon^{-j\left(\omega t - \frac{\pi}{2} - \theta\right)}}{2} \tag{62}$$

Equations (61) and (62) are true algebraic equations for a sinusoidally varying current and any mathematical operation may be performed on them. Complex, polar or exponential expressions, when used for an alternating current, give the revolving vector, which represents the current, and require proper interpretation. They are symbolic, and mathematical operations cannot be performed on them without first determining whether they give correct results.

Representation of a Vector Which Rotates with a Uniform Angular Velocity and Shrinks in Magnitude Logarithmically with Time.

$$\begin{aligned} \bar{x} &= A\epsilon^{-at}(\cos \omega t \pm j \sin \omega t) \\ &= A\epsilon^{-at}\epsilon^{\pm j\omega t} \end{aligned}$$

is a revolving vector which rotates with a uniform angular velocity ωt and shrinks logarithmically with time at a rate determined by the constant exponent a. A is a constant and is equal to the length of the vector when time t is zero.

If the angle ωt is measured from the horizontal axis, the projection on a vertical axis of the revolving vector $\bar{x} = A\epsilon^{-at}\epsilon^{\pm j\omega t}$ represents a sinusoidally oscillating quantity whose successive maximum values decrease logarithmically with time. The current produced by the discharge of a condenser through an inductive circuit of low resistance may be represented in this way.

CHAPTER II

ALTERNATING CURRENTS

Direct Current or Voltage.—A direct current or voltage is unidirectional and is practically steady and non-pulsating in magnitude.

Oscillating Current or Voltage.—An oscillating current or voltage is one which alternately increases and decreases in magnitude with respect to time.

Periodic Current or Voltage.—A periodic current or voltage is one which oscillates and the values of which recur for equal increments of time.

Pulsating Current or Voltage.—A pulsating current or voltage is one that is periodic and the values of which are always of the same sign, *i.e.*, either positive or negative.

Alternating Current or Voltage.—An alternating current or voltage is one that is periodic and has alternately positive and negative values.

Symmetrical Alternating Current or Voltage.—A symmetrical alternating current or voltage is one in which all values separated by half a period have the same magnitudes but opposite signs.

Instantaneous Value.—The instantaneous value of a current or voltage is its value at any given instant of time t.

Cycle.—A cycle of an alternating current or voltage is one complete set of positive and negative values of the current or voltage. These values repeat themselves at regular intervals.

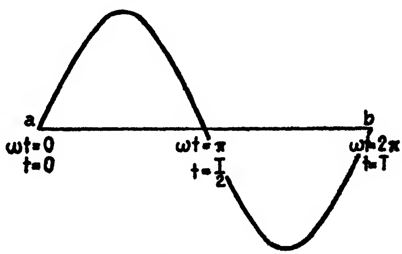


FIG. 10.

Periodic Time or Period.—The periodic time or period of an alternating current or voltage is the time required for it to pass through one complete cycle of values. It is expressed in seconds and is

ordinarily denoted by T. *A cycle is represented in Fig.* 10 by the portion of the wave between a and b.

Frequency.—The number of cycles passed through by an alternating current or voltage in a second is its frequency. It is denoted by the letter f. For example, a 60-cycle circuit is one in which the current and voltage pass through 60 complete cycles per second. Its frequency and periodic time are, respectively,

$$f = 60 \text{ cycles per second}$$

and

$$T = \frac{1}{f} = \frac{1}{60} = 0.01667 \text{ second}$$

The frequencies commonly used in America are 60 and 25 cycles per second, although 50 cycles per second is also used. Abroad, 50 cycles per second is common and there are a few installations for railway work with frequencies lower than 25 cycles per second. In general, frequencies as low as 25 cycles per second are unsatisfactory for lighting, on account of the noticeable flicker in the lights produced in many cases by so low a frequency. Sixty cycles per second is almost universally used for lighting, although 50 cycles per second is perfectly satisfactory. Twenty-five cycles per second was formerly used for long-distance power transmission and for installations where power was the primary object. However, due to the better design of apparatus and also to a better understanding of the problems of long-distance power transmission, 60 cycles per second is now used for both long-distance power transmission and ordinary power and lighting. In general, 60-cycle apparatus is lighter and cheaper than 25-cycle apparatus. The difference in weight and cost between 60-cycle and 25-cycle apparatus is essentially the same as between high-speed and low-speed direct-current motors and generators. In speaking of frequency, cycles per second is always understood. For this reason, it is common engineering practice to say 60 cycles instead of 60 cycles per second.

Wave Shape or Wave Form.—The shape of the curve obtained when the instantaneous values of a voltage or a current are plotted as ordinates against time as abscissas is its wave shape or

wave form. The abscissa for one cycle is taken as 2π radians or 360 degrees, and corresponds to one complete revolution of the armature of a two-pole alternator.

Two alternating currents or voltages are said to have the same wave form when their ordinates, at corresponding positions in degrees measured from the zero points of the waves, bear a constant ratio to each other.

Simple Harmonic Current or Voltage.—The form of periodic current or voltage most easily dealt with mathematically is one whose instantaneous values follow one another according to the law of sines. In general, this wave form is most desirable from the standpoint of the generation, transmission and utilization of power. The sine-wave form of a periodic voltage, *i.e.*, a sinusoidal voltage, is given by equation (1), where the time t is reckoned from the instant when the voltage is zero and increasing in a positive direction.

$$v = V_m \sin 2\pi \frac{t}{T} = V_m \sin 2\pi f t = V_m \sin \omega t \qquad (1)$$

V_m is the maximum value of the wave or its *amplitude.* T is the duration of one complete cycle, *i.e.*, the periodic time, and f is the number of complete cycles per second, *i.e.*, the frequency.

A simple harmonic or sinusoidal wave is positive during one half of each cycle and negative during the other half of the cycle. A simple harmonic current is plotted in Fig. 10 with angles as abscissas. Figure 10 is its wave form. The abscissas might equally well have been time, 2π radians corresponding to the time of one complete cycle, *i.e.*, to $T = \frac{1}{f}$ second. The abscissas are indicated in both these ways in the figure.

Alternating-current Calculations Based on Sine Waves.—Alternating-current calculations are commonly based on the assumption of sinusoidal waves of current and voltage. Where the waves are not sinusoidal, it is possible to resolve them into component sinusoidal waves, known as the *fundamental* and *harmonics*, having frequencies which are 1, 2, 3, 4, 5, 6 etc. times the frequency of the circuit. The components which have frequencies equal to 2, 4, 6 etc. times the frequency of the circuit, *i.e.*, the so-called *even* harmonics, are not present in

symmetrical waves. (See Chapter IV.) The effects of the different component sinusoidal waves may be determined separately and then combined.

Since any periodic alternating current can be analyzed into a series of simple sinusoidal waves of definite frequencies, it is desirable to discuss in considerable detail the conditions which hold for circuits carrying simple sinusoidal or simple harmonic currents.

A periodic current or voltage need not follow the simple sine law, but its instantaneous values must be some function of time and it must go through a complete cycle in $\frac{1}{f}$ second. Its positive and negative loops need not be symmetrical, but on account of the way in which periodic waves are generated in rotating machinery, the positive and negative loops for such machinery are symmetrical with respect to the axis of time. A periodic current which does not follow the simple sine law is shown in Fig. 11.

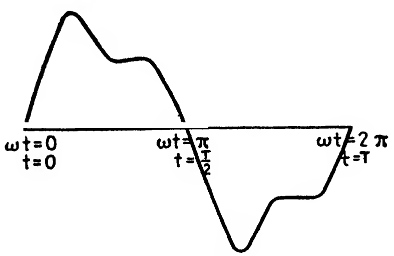


Fig. 11.

Phase.—Time need not be reckoned from the instant when a current or voltage wave passes through zero and in many cases it is not so reckoned. If time is considered zero at a point α radians (equivalent to $\frac{\alpha}{2\pi} \times \frac{1}{f}$ second) after the wave has passed through zero increasing in a positive direction, equation (1) becomes

$$v = V_m \sin(\omega t + \alpha) \tag{2}$$

The angle α is known as the *phase angle of the wave.* It indicates the number of radians between the point where the wave is zero and increasing in a positive direction and the point from which time is reckoned. The angle may be either positive or negative. When positive, it is an angle of lead and the voltage goes through zero increasing in a positive direction before time t is zero. When the angle is negative, it is an angle of lag and the voltage goes through zero increasing in a positive direction after time t is zero.

Equations (3) and (4) represent a voltage and a current of the same frequency, both of which are sinusoidal, but which do not go through their zero values at the same instant.

$$v = V_m \sin(\omega t + \alpha_1) \tag{3}$$
$$i = I_m \sin(\omega t + \alpha_2) \tag{4}$$

Although both waves have the same frequency, they are not in phase. Their difference in phase is the difference between their phase angles. When the phase angles are angles of lead as in equations (3) and (4), the angle of lead of v with respect to i is the phase angle of v minus the phase angle of i or $(\alpha_1 - \alpha_2)$. The angle of lead of i with respect to v is the phase angle of i minus the phase angle of v or $(\alpha_2 - \alpha_1)$. The corresponding angles of lag are $(\alpha_2 - \alpha_1)$ for v with respect to i and $(\alpha_1 - \alpha_2)$ for i with respect to v. When α_2 is less than α_1, $(\alpha_1 - \alpha_2)$ is positive and v actually leads i. When α_2 is greater than α_1, $(\alpha_1 - \alpha_2)$ is negative and $(\alpha_2 - \alpha_1)$ is positive. In this case v leads i by a negative angle or lags i by a positive angle. Since a negative angle of lead is equivalent to a positive angle of lag, when $(\alpha_1 - \alpha_2)$ is negative v actually lags i. A negative angle of lead is always equivalent to an actual angle of lag, and, similarly, a negative angle of lag is always equivalent to an actual angle of lead.

Although the phase angles in equations (3) and (4) should be expressed in radians to be mathematically correct, it is usually more convenient to express them in degrees. When this is done, both ωt and α must be reduced to the same units, *i.e.*, radians or degrees, before they can be added.

Generation of an Alternating Electromotive Force.—The electromotive force generated or induced in a coil through which flux is varying is equal to the rate of change of flux through the coil with respect to time multiplied by the number of turns in the coil, *i.e.*, it is equal to the time rate of change of flux linkages for the coil. The flux linkages are equal to the flux through the coil multiplied by the number of turns with which this flux links. If e represents the instantaneous voltage induced in a coil by a change in the flux linking it,

$$e = -N\frac{d\varphi}{dt} \tag{5}$$

The time rate of change of flux through the coil is $\frac{d\varphi}{dt}$ and $N\frac{d\varphi}{dt}$ is the time rate of change of flux linkages with the coil. The voltage e in equation (5) is a voltage rise. When the minus sign is omitted it is a voltage fall or drop. (For the significance of rise and fall applied to a voltage see page 68.)

The electromotive forces induced or generated in the armature turns of a direct-current generator are alternating, but they are rectified with respect to the external circuit by means of the commutator.

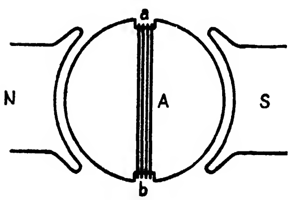


FIG. 12.

Figure 12 shows a simple two-pole alternator with a revolving armature and a single armature coil.

N and S are the poles. A is the armature which carries a winding placed in two diametrical slots a and b. The terminals of this winding, in an actual machine, would be brought out to two insulated slip rings mounted on the armature shaft. The current would be taken from these slip rings by means of brushes. If the poles of the alternator are so shaped that the flux through the coil varies as the cosine of the angular displacement of the coil from the vertical position (see Fig. 12), a sinusoidal wave of electromotive force is generated. In this case

$$e = -N\frac{d\varphi}{dt} = -N\frac{d}{dt}\varphi_m \cos \omega t$$

$$= \omega N \varphi_m \sin \omega t \qquad (6)$$

where N is the number of turns in the armature winding and ω is the angular velocity of the armature in radians per second. The maximum value of the flux through the coil is φ_m. In case the alternator has more than two poles, there are as many similar winding elements as there are pairs of poles. These are spaced at a distance apart on the armature equal to the distance between two adjacent poles of like sign. In other words, they are spaced 360 *electrical* degrees or 2π *electrical* radians apart. A movement of the armature of a multipolar alternator through 2π electrical radians is equivalent, so far as the generation of voltage is concerned, to a movement of the armature

of a two-pole alternator through 2π actual or *space* radians. The angular velocity of the armature in electrical radians is always equal to $\omega = 2\pi f$ where f is the frequency of the voltage generated. For a two-pole machine, frequency is equal to the revolutions made by its armature per second. For a multipolar alternator it is equal to revolutions per second multiplied by the number of pairs of poles.

$$f = (\text{revolutions per second}) \times (\text{number of pairs of poles}) \qquad (7)$$

Since the elements of the armature winding for a multipolar alternator are 2π electrical radians apart, they lie, at each instant, in the same relative position with respect to north and south poles. The voltages generated in them are, therefore, in phase. They are also equal, since each element of the winding contains the same number of turns. Since the voltages are both equal and in phase, the elements may be connected either in series or in parallel. Series connection, however, is the more common as high voltage is usually desired. The standard voltages commonly used for alternators are from 2300 to 14,000 volts. The frequencies most commonly used are 60 cycles and 25 cycles.

The armature windings of alternators, as actually built, are not placed in a single pair of slots per pair of poles, as shown in Fig. 12, but are distributed among a number of pairs of slots for each pair of poles and are thus made to cover a considerable portion of the armature surface. Spreading out the winding in this way increases the output obtainable from an armature of given size and also improves the wave form. The armature winding of a single-phase alternator covers about two-thirds of the armature periphery. The armature windings of a polyphase alternator cover the entire armature surface, but the winding for any one phase covers only a portion of it.

Commercial alternators, except in *very* small sizes, are universally built with stationary armatures and revolving fields. (See Fig. 86, page 283.) With such construction, the more complicated part, *i.e.*, the armature winding, is stationary and thus is relieved from all stresses except those caused by the current. Moreover, no sliding contacts are required to collect the high-voltage armature current. The only necessary sliding

contacts are those for the field excitation. These carry a comparatively small current at low voltage.

Strength of Current.—The strength of an alternating current is defined in terms of its heating effect. An alternating current and a steady current are said to be equal if they produce heat at equal rates when they flow through equal resistances. This assumes that the distribution of current over the cross section of the conductor is the same in both cases.

Let i be the instantaneous value of the alternating current, *i.e.*, its strength at any instant of time t, and let T denote the duration of a complete cycle. The rate of heating in a given constant resistance r at any instant is i^2r, and the average rate of heating during a cycle is

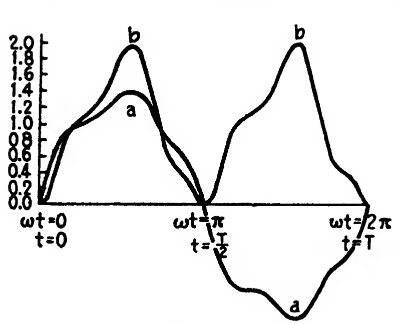


Fig. 13.

$$\frac{1}{T}\int_0^T i^2r\,dt = \frac{r}{T}\int_0^T i^2dt \quad (8)$$

A current wave (curve a) is plotted in Fig. 13. Curve b on this figure is obtained by plotting the squared ordinates of curve a. The integral $\frac{r}{T}\int_0^T i^2dt$ is the mean ordinate of the curve b multiplied by r, the resistance of the circuit.

The expression $\frac{1}{T}\int_0^T i^2dt$ is the mean value of i^2, *i.e.*, the mean square of the instantaneous values of the current. Denoting this by (mean i^2), the average rate of heating produced by a periodic current is (mean i^2) $\times r$. For a steady current I, the rate of heating as given by Joule's law is I^2r. Hence, in order that the heating effect of an alternating and of a steady current shall be equal for equal resistances, (mean i^2) must equal I^2 or $\sqrt{(\text{mean } i^2)} = I$. From this it follows directly that a periodic and a steady current are equal when $\sqrt{(\text{mean } i^2)} = I$, and, according to this definition, the rate of heating by a periodic current as well as by a steady or direct current follows Joule's law.

Ampere Value of an Alternating Current.—An alternating current is said to have a strength of one ampere when the average rate at which it produces heat in a given resistance is equal to the rate at which heat is produced by a steady or direct current of one ampere when passing through an equal resistance. Obviously, according to this definition, the ampere value of an alternating current is given by

$$\sqrt{\frac{1}{T}\int_0^T i^2dt} = I \tag{9}$$

Volt Value of an Alternating Voltage.—The corresponding definition of a volt is that potential difference which maintains one ampere of alternating current, as just defined, through a non-inductive resistance of one ohm. Obviously, an alternating voltage, as well as an alternating current, is measured by the square root of its average square value.

Advantages of Defining the Strength of Alternating Currents by the Square Roots of Their Average Square Values.—The advantages of this method of defining the strength of alternating currents are:

(*a*) It makes it possible to measure any alternating current, of whatever wave form, by its heating effect;

(*b*) It makes the electrodynamometer available for the measurement of all alternating currents and also for power measurements;

(*c*) It brings alternating currents under the same laws of heating as steady or direct currents;

(*d*) It avoids the necessity of a factor in the expression for power which would be different for each wave form.

The square root of the mean square of the instantaneous values of an alternating current or its root-mean-square value, usually abbreviated into r.m.s. value, is known as its *effective* value.

That this method of defining an alternating current gives a different result from that which would be obtained by defining the value of the current as the average of its instantaneous values is obvious. The average over a complete cycle, for all symmetrical waves, would be zero. For this reason, when the average value of an alternating current is mentioned, the average value over half a cycle is always understood.

The average value of an alternating current is given by

$$I_{av.} = \frac{2}{T}\int_0^{\frac{T}{2}} i dt \tag{10}$$

The root-mean-square value is given by

$$I_{r.m.s.} = \sqrt{\frac{1}{T}\int_0^{T} i^2 dt} \tag{11}$$

These are equal only when i is constant, *i.e.*, for steady currents. The average value of the current is equal to the area enclosed by either loop of the curve a, Fig. 13, page 36, divided by the base of the loop. The root-mean-square value of the current is the square root of the ratio of the area under the curve b to its base.

Relation between the Root-mean-square and Average Values for Simple Harmonic or Sinusoidal Currents.—The relation between the root-mean-square and the average values of a simple harmonic current is easily determined.

$$I_{r.m.s.}^2 = \frac{1}{T}\int_0^T i^2 dt = \frac{1}{T}\int_0^T \left(I_m \sin\frac{2\pi t}{T}\right)^2 dt$$

$$= \frac{I_m^2}{2}$$

$$I_{r.m.s.} = \frac{I_m}{\sqrt{2}} = 0.707 I_m \tag{12}$$

$$I_{av.} = \frac{2}{T}\int_0^{\frac{T}{2}} i\, dt = \frac{2}{T}\int_0^{\frac{T}{2}} I_m \sin\frac{2\pi t}{T}\, dt$$

$$= \frac{2}{\pi} I_m = 0.637 I_m \tag{13}$$

Hence

$$\frac{I_{av.}}{I_{r.m.s.}} = \frac{0.637}{0.707} = 0.901 \tag{14}$$

$$\frac{I_{r.m.s.}}{I_{av.}} = \frac{0.707}{0.637} = 1.11 \tag{15}$$

This relation between the average and root-mean-square values of an alternating current may be made clearer by the statement that if a simple harmonic current of 1 ampere were flowing

in a circuit, the average of its instantaneous values in one direction would be only 0.901 ampere steady current. The number of coulombs in time t would be $0.901t$, or, in general, for I amperes $0.901It$ coulombs. If a simple harmonic current could be commutated at each $\frac{T}{2}$ second, *i.e.*, every half cycle, the weight of copper deposited by it per second would be only 0.901 of that deposited by a steady current of the same number of amperes.

Form Factor.—The ratio of the root-mean-square or effective value of a symmetrical alternating current or voltage to its average value is its form factor. (See page 114.) For a simple harmonic wave this is 1.11.

Measurement of the Effective or Root-mean-square Value of a Current or Voltage.—The effective or root-mean-square value of an alternating current of the frequency ordinarily used for power may be measured by an electrodynamometer. Such an instrument, in its simplest form, consists of two concentric coils, one slightly smaller than the other. The smaller coil is mounted inside of the larger in such a manner that it may turn about a diameter which coincides with a diameter of the larger coil. The movement of the smaller coil is resisted by two spiral springs attached to its shaft. These two springs also serve to connect the coil in circuit. When no current is passing through the two coils, the springs hold the plane of the coils at an angle of about 45 degrees with each other.

If the two coils are connected in series and a current is passed through them, a torque is developed on the coils which tends to make the movable coil turn to enclose the maximum flux. For any relative position of the coils, this torque is proportional to the product of their ampere turns. Since they are in series and carry the same current, the torque is also proportional to the square of the current. The movable coil deflects until the torque exerted on it, due to the current, is just balanced by the opposing torque of the control springs. The direction in which this coil moves depends upon the relative direction of the current in the coils. The relative direction of the current should be such as to make the movable coil swing so as to increase the angle it makes with the other coil. This gives the maximum scale range. The greatest sensitivity occurs when the two coils

are at right angles. If the current reverses, the torque exerted on the coils does not reverse, since the relative direction of the currents in the two coils does not change.

When such an instrument is connected in an alternating-current circuit, the torque exerted on the coils is proportional, at every instant, to the square of the instantaneous current. The average torque is proportional to the average square of the instantaneous current or to the square of the effective or root-mean-square current. By attaching a pointer to the movable coil and providing the instrument with a suitably graduated scale, it may be made to indicate effective current. Since, for any given position of the coils, the torque exerted on them varies as the square of the current, the scale cannot be uniform. In general, it is more or less cramped at the ends and open in the middle.

Many refinements are made in the instrument as actually constructed to increase its sensitivity and to make its scale more uniform.

Since the control springs must serve to carry the current to and from the movable coil, the use of the electrodynamometer type of ammeter is limited to the measurement of very small currents, a tenth of an ampere or less, unless the movable coil is shunted with an inductive shunt. For measuring larger currents, the movable coil is generally replaced by a soft-iron vane. The torque acting to deflect this vane is proportional to its magnetic moment and to that component of the field produced by the fixed coil which is *perpendicular* to the axis of the vane. If the vane is operated at low flux density and is made of suitable iron, its magnetic moment is proportional to the component of the field which lies *along* its axis. It is obvious that for any fixed position of the vane, the torque acting to deflect it is proportional to the square of the current in the fixed or field coil. If the current reverses, both the field and the magnetic moment of the vane reverse. The direction of the torque, therefore, remains unchanged. When an instrument arranged in this way is connected in series with an alternating-current circuit, the vane takes up a position which is determined by the average square value of the current. The limit of deflection is reached when the vane is perpendicular to the plane of the coil. To

extend the range of deflection, the coil of the iron-vane type of instrument is inclined 45 degrees with the axis about which the vane turns.

If either of the two types of ammeter is wound with many turns to measure small currents, it may be used also as a voltmeter. In this case a suitable non-inductive resistance is connected in series with it before placing the instrument across the terminals of the circuit whose potential is to be measured. Since the current in the instrument is then proportional to the voltage at every instant, the instrument may, by suitable calibration, be made to indicate effective or root-mean-square voltage.

Nearly all alternating-current ammeters for power frequencies, except those for measuring small currents, are of the iron-vane type. The best voltmeters are of the electrodynamometer type. The electrodynamometer type of instrument is much superior to instruments having iron vanes, but its use is limited to the measurement of small currents. For high frequencies other types of instruments must be used.

Representation of Simple Harmonic Currents and Voltages by Revolving Vectors.—The simple harmonic current

$$i = I_m \sin (\omega t + \theta)$$

is represented graphically by the curve of sines plotted in Fig. 14. In this figure, angles are plotted as abscissas and currents as ordinates. The abscissas of the curve are also marked with the corresponding values of time t. The ordinates of this curve may be obtained by projecting a vector $0A$, revolving in a positive direction, on the fixed reference line $0Y$. The vector $0A$ must be equal in length to the maximum value of the current and it must revolve at a constant angular velocity of $\omega = 2\pi f$ radians per second. Angles are reckoned from the horizontal reference axis.

The counter-clockwise direction of rotation, as was stated in Chapter I, is always considered positive for rotating vectors. Positive angles, therefore, are measured in a counter-clockwise direction and are angles of lead. Negative angles or angles of lag are measured in a clockwise direction. When there are several vectors, it is sometimes convenient to fix the vectors and

rotate the axes of reference. When this is done, a clockwise rotation of the axes is positive.

The vector $0A$, Fig. 14, makes one revolution in a time $T = \frac{1}{f}$ second, the periodic time of the current, and occupies a position perpendicular to the axis $0Y$, the axis on which the projections are taken, at a time determined by the condition $(\omega t + \theta) = 0$. Since the instantaneous values of the current are determined from the projections of the vector $0A$ on the perpendicular axis $0Y$, this vector $0A$ may be considered to represent the current $i = I_m \sin (\omega t + \theta)$. The phase of the current is so determined that, at a time $t = 0$, the vector $0A = I_m$ makes a positive angle

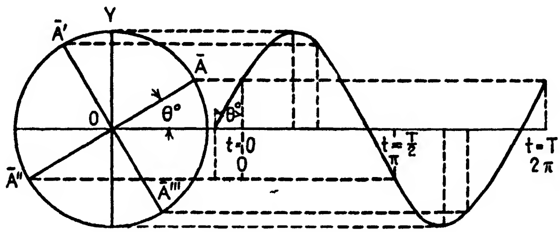


FIG. 14.

θ, its phase angle, with the horizontal reference axis from which angles are reckoned. The rotating vector exactly represents the current in both magnitude and phase, since its projection on the vertical axis $0Y$ at any instant represents the magnitude of the current at that instant and the angle $(\omega t + \theta)$ which it makes with the horizontal axis represents the phase of the current at that instant.

Vector Notation Applied to Alternating Currents and Voltages. The vector $0A$, Fig. 14, which may be considered to represent a sinusoidal alternating current, may be expressed in either rectangular or polar coördinates. At any instant of time, such as t, the vector $0A$ stands $(\omega t + \theta)$ radians from the horizontal axis from which angles are measured. In rectangular coördinates, the vector is

$$\bar{I}_m = I_m\{\cos (\omega t + \theta) + j \sin (\omega t + \theta)\} \tag{16}$$

where $\{\cos (\omega t + \theta) + j \sin (\omega t + \theta)\}$ is the complex operator which produces a rotation of ω radians per second.

In terms of polar coördinates, the expression for the vector is

$$\bar{I}_m = I_m\underline{|(\omega t + \theta)} \tag{17}$$

The vector may also be expressed by the use of the exponential operator. Making use of this operator, the expression for the vector is

$$\bar{I}_m = I_m\epsilon^{j(\omega t+\theta)} \tag{18}$$

The expressions given for the current by equations (16), (17) and (18) are all symbolic since they give the revolving vector and not its projection on the vertical axis.

In most cases, when dealing with steady conditions in a circuit, *i.e.*, not transient conditions, the revolving vectors may be considered fixed in the positions they occupy at some arbitrarily chosen instant of time which is frequently taken as zero time. Considered in this way, the vector expressions for the current $i = I_m \sin(\omega t + \theta)$ are

$$\bar{I} = I_m(\cos\theta + j\sin\theta) \tag{19}$$

$$= a + jb$$

$$\tan\theta = \frac{b}{a}$$

$$\bar{I} = I_m\underline{|\theta} \tag{20}$$

$$\bar{I}_m = I_m\epsilon^{j\theta} \tag{21}$$

Equations (19), (20) and (21) are respectively the complex, the polar and the exponential expressions for the current

$$i = I_m \sin(\omega t + \theta)$$

Since effective values of current and voltage are wanted, it is customary in writing vector expressions for currents and voltages to use effective values instead of maximum values. Maximum values are necessary only when the instantaneous values of the current or voltage are to be found from the projections of the vectors on the vertical axis.

Since the addition or subtraction of two or more vectors expressed in rectangular coördinates simply involves adding or subtracting separately the components along the two axes, the complex expressions for currents and voltages are particularly useful when currents or voltages are to be added or subtracted. They also may be used for multiplication and division.

Since multiplication of vectors involves multiplication of magnitudes and addition of phase angles, and division of vectors involves division of magnitudes and subtraction of phase angles (see page 16), the polar and exponential expressions are best adapted for multiplication and division. They cannot be used for addition or subtraction but for these operations they may be converted readily to the complex forms. They are useful in finding roots, powers etc. of a vector.

A sinusoidal alternating current may be exactly represented by two conjugate oppositely rotating vectors. (See page 26.) Expressed in this way,

$$i = I_m \sin(\omega t + \theta) = \frac{I_m \epsilon^{j(\omega t - \frac{\pi}{2} + \theta)} + I_m \epsilon^{-j(\omega t - \frac{\pi}{2} + \theta)}}{2} \tag{22}$$

$$i = I_m \cos(\omega t + \theta) = \frac{I_m \epsilon^{j(\omega t + \theta)} + I_m \epsilon^{-j(\omega t + \theta)}}{2} \tag{23}$$

The second of these is the simpler and, therefore, the more useful.

The expressions given in equations (22) and (23) are true algebraic equations, and, therefore, any mathematical operation may be performed on them. They are particularly useful in certain of the more complicated alternating-current problems. They have been used to advantage in the study of certain transient phenomena in rotating alternating-current machinery.

Another symbolic expression which is used for an alternating current is

$$i = (Re) I_m \epsilon^{j(\omega t + \theta)} \tag{24}$$

where the symbol (Re) is an operator which takes the real part of the revolving vector $I_m \epsilon^{j(\omega t + \theta)}$, *i.e.*, its projection on the axis of reals. Equation (24) gives $i = I_m \cos(\omega t + \theta)$. The angle θ in the exponent may be avoided by replacing the magnitude I_m by the vector $\bar{I}_m$, where

$$\bar{I}_m = I_m (\cos\theta + j \sin\theta)$$

This takes care of the phase angle θ. Making this change, the symbolic expression for a current which is given by equation (24) becomes

$$i = (Re) \bar{I}_m \epsilon^{j\omega t} \tag{25}$$

A similar expression holds for a voltage.

Since equation (25) is symbolic, mathematical operations must not be performed on it without first determining whether the results produced have real significance. Any mathematical operation may be performed on equation (25) provided the real part of the resulting expression is the same as the expression obtained by performing the same operation on the real part of equation (25).

The chief operations that can be performed on equation (25) for a current are differentiation and integration. When other operations are to be performed, the expression for the current can be converted into a suitable form.

The symbolic form for a current given by equation (25) is particularly useful in the study of electrical networks.

Addition of Currents.—When considering several harmonic currents and voltages of the same frequency, each must be

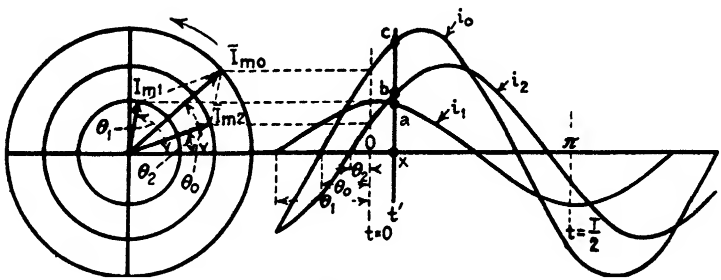


FIG. 15.

represented in both magnitude and phase by a revolving vector of proper length and position, all vectors being drawn from the same point.

Consider a circuit consisting of two branches in parallel carrying simple harmonic currents i_1 and i_2 given by the following equations:

$$i_1 = I_{m1} \sin (\omega t + \theta_1) \tag{26}$$

$$i_2 = I_{m2} \sin (\omega t + \theta_2) \tag{27}$$

These two currents and their resultant i_0 are plotted in Fig. 15.

Take the axis of reference as the axis of reals (see page 6, Chapter I) and consider the vectors at the instant $t = 0$. At this instant I_{m0}, I_{m1} and I_{m2} make angles θ_0, θ_1 and θ_2 with the

axis of reals. Their complex expressions at this instant are, therefore,

$$
\begin{aligned}
\bar{I}_{m0} &= I_{m0}\,(\cos\theta_0 + j\sin\theta_0) = a_0 + jb_0 \\
\bar{I}_{m1} &= I_{m1}\,(\cos\theta_1 + j\sin\theta_1) = a_1 + jb_1 \\
\bar{I}_{m2} &= I_{m2}\,(\cos\theta_2 + j\sin\theta_2) = a_2 + jb_2 \\
\bar{I}_{m0} &= I_{m1} + I_{m2} \\
a_0 + jb_0 &= (a_1 + jb_1) + (a_2 + jb_2) \\
&= (a_1 + a_2) + j\,(b_1 + b_2) \\
\bar{I}_{m0} &= \sqrt{a_0^2 + b_0^2} = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \\
&= \sqrt{(I_{m1}\cos\theta_1 + I_{m2}\cos\theta_2)^2 + (I_{m1}\sin\theta_1 + I_{m2}\sin\theta_2)^2} \\
&= \sqrt{(\Sigma_1^2\, I_m \cos\theta)^2 + (\Sigma_1\, I_m \sin\theta)^2} \qquad (28) \\
\tan\theta_0 &= \frac{b_1 + b_2}{a_1 + a_2} \\
&= \frac{I_{m1}\sin\theta_1 + I_{m2}\sin\theta_2}{I_{m1}\cos\theta_1 + I_{m2}\cos\theta_2} \\
&= \frac{\Sigma_1^2\, I_m \sin\theta}{\Sigma_1^2\, I_m \cos\theta} \qquad (29)
\end{aligned}
$$

Equations (28) and (29) are not limited to two currents or in general to two revolving vectors. They may be extended to apply to any number. If there are k currents of the same frequency to be added,

$$
\begin{aligned}
\bar{I}_{m0} = a_0 + jb_0 &= (a_1 + a_2 + \cdots + a_k) + j(b_1 + b_2 + \cdots + b_k) \\
\tan\theta_0 &= \frac{b_1 + b_2 + \cdots + b_k}{a_1 + a_2 + \cdots + a_k} \\
\bar{I}_{m0} &= \Sigma_1^k\, I_m \cos\theta + j\,\Sigma_1^k\, I_m \sin\theta \\
I_{m0} &= \sqrt{(\Sigma_1^k\, I_m \cos\theta)^2 + (\Sigma_1^k\, I_m \sin\theta)^2} \qquad (30) \\
\tan\theta_0 &= \frac{\Sigma_1^k\, I_m \sin\theta}{\Sigma_1^k\, I_m \cos\theta} \qquad (31)
\end{aligned}
$$

Since the relations existing among the maximum values of rotating vectors and also among their phase angles are independent of the instant of time considered, it is not necessary to take $t = 0$ when adding currents or voltages. The value of t which gives the simplest and easiest solution should be chosen. Usually this is $t = 0$, although it is better, in many cases, to give t such a value that one of the vectors lies along the axis of reals. When the solution of a problem involves both currents

and voltages and there is a common vector, such as current in a series circuit or voltage in a parallel circuit, this common vector may well be taken for the axis of reals. For example, in a circuit consisting of a number of branches in parallel the same voltage is impressed across each. It is common to the currents in the branches. In general, it should be taken as the axis of reals. In the case of a series circuit, the current is the same in all parts. In general, for a series circuit, the current should be taken for the axis of reals.

In most problems arising in alternating-current work, interest centers on the effective values of currents and voltages, *i.e.*, root-mean-square values. Maximum values are seldom used or desired. Since root-mean-square values and maximum values are proportional ($I_m = \sqrt{2}\, I_{r.m.s.}$), it is customary to write the vector expressions for all alternating currents and voltages in terms of their root-mean-square or effective values.

If currents or voltages are to be subtracted, a method similar to that outlined above for the addition of currents should be followed.

An Example of Addition of Currents.—A load on a certain line consists of an induction motor in parallel with a synchronous motor. Since the motors are in parallel, the voltages across their terminals must be equal at every instant. The vectors representing the voltages impressed across the terminals of the motors must, therefore, be in phase and must also be equal in magnitude. The induction motor takes a current of 100 amperes (root-mean-square), which lags the impressed voltage by 30 degrees. The current taken by the synchronous motor is 50 amperes. This leads the impressed voltage by 60 degrees. What is the resultant current which must be supplied to these motors in parallel and what is its phase with respect to the voltage impressed across their terminals?

Consider time t zero when the instantaneous voltage is zero. The expressions for the instantaneous voltage and currents then are

$$v = V_m \sin \omega t$$
$$i_i = 100\sqrt{2} \sin (\omega t - 30°)$$
$$i_s = 50\sqrt{2} \sin (\omega t + 60°)$$

At the instant of time $t = 0$, the voltage vector lies on the horizontal axis. Take this as the axis of reals. Then the complex expressions for the vectors which represent the voltage and currents, using root-mean-square values, are

$$
\begin{aligned}
\bar{V} &= V(1 + j0) \\
\bar{I}_i &= 100(\cos 30° - j \sin 30°) \\
&= 86.6 - j50 \\
\bar{I}_s &= 50(\cos 60° + j \sin 60°) \\
&= 25.0 + j43.3 \\
\bar{I}_0 &= \bar{I}_i + \bar{I}_s \\
&= (86.6 - j50) + (25.0 + j43.3) \\
&= 111.6 - j6.7 \\
I_0 &= \sqrt{(111.6)^2 + (6.7)^2} \\
&= 111.9 \text{ amperes (r.m.s.)} \\
\tan \theta_0 &= \frac{-6.7}{111.6} = -0.0600 \\
\theta_0 &= -3.44 \text{ degrees}
\end{aligned}
$$

The resultant current I_0 is 111.9 amperes and it lags the voltage impressed across the motors by 3.44 degrees. In other words, it goes through its cycle 3.44 degrees or $3.44 \times \frac{1}{360 \times f}$ seconds later than the voltage, f being the frequency of the circuit.

Another Example.—A two-pole, three-phase, 60-cycle alternator has three exactly similar windings on its armature, spaced 120 degrees from one another and each containing 100 turns. Two of these windings are connected in series with each other in such a way that the voltage between their free terminals is equal to the difference between their induced voltages. If these two windings in series are connected to a constant non-inductive resistance of 10 ohms, what are the maximum and effective values of the current in the resistance? How much power is absorbed by the resistance? What is the phase relation of the current in the resistance to the voltage impressed across its terminals? What is the phase relation between the voltage across the two windings in series and the voltage induced in each winding? The field flux of the alternator per pole is 10^6 maxwells. This is distributed in such a way that the flux linking each armature winding varies sinusoidally with time.

The voltage induced in any armature winding is

$$e = -N\frac{d\varphi}{dt}$$

where φ is the flux linking the winding at the time t and N is the number of turns in the winding. According to the assumption that the flux linking each armature winding varies sinusoidally with time,

$$\varphi = \varphi_m \sin \omega t$$

where ω is the angular velocity of the armature in electrical radians per second and is equal to 2π times the frequency of the voltage induced in it. For a two-pole alternator, the frequency f is equal to the speed of its armature in revolutions per second.

$$\begin{aligned} e &= -N\frac{d}{dt}\varphi_m \sin \omega t \\ &= -\omega N \varphi_m \cos \omega t \end{aligned}$$

The maximum value of this voltage is obviously

$$E_m = \omega N \varphi_m$$

The effective or root-mean-square value is

$$\begin{aligned} E &= \frac{E_m}{\sqrt{2}} = \frac{\omega}{\sqrt{2}} N \varphi_m = \frac{2\pi f}{\sqrt{2}} N \varphi_m \\ &= 4.44\ N f \varphi_m \text{ abvolts} \\ &= 4.44\ N f \varphi_m 10^{-8} \text{ volts} \end{aligned}$$

The voltage induced in each winding of the alternator is, therefore,

$$\begin{aligned} E &= 4.44 \times 100 \times 60 \times 10^{6} \times 10^{-8} \\ &= 266.4 \text{ volts effective} \end{aligned}$$

Let the armature windings be numbered 1, 2, 3 and assume that the direction of rotation of the armature is such that the phase order is also 1, 2, 3, *i.e.*, the voltages induced in the armature winding go through their cycles in the order 1, 2, 3. The three voltages must be 120 degrees apart in time phase since the armature windings are spaced 120 degrees apart on the armature. Let the windings 1 and 2 be the ones connected to the resistance. Take the voltage induced in phase 1 as along the axis of reals. The vector expressions for the three voltages are then

$$\begin{aligned}
\bar{E}_1 &= E_1(\cos 0° - j \sin 0°) \\
&= 266.4(1 - j0) \\
&= 266.4 - j0 \\
\bar{E}_2 &= E_2(\cos 120° - j \sin 120°) \\
&= 266.4(-0.5 - j\,0.866) \\
&= -133.2 - j230.7 \\
\bar{E}_3 &= E_3(\cos 240° - j \sin 240°) \\
&= 266.4\,(-0.5 + j\,0.866) \\
&= -133.2 + j230.7
\end{aligned}$$

The voltage between the terminals of windings 1 and 2, connected in such a way that their voltages subtract, is

$$\begin{aligned}
\bar{E}_0 = \bar{E}_1 - \bar{E}_2 &= (266.4 - j\,0) - (-133.2 - j\,230.7) \\
&= 399.6 + j\,230.7 \\
E_0 &= \sqrt{(399.6)^2 + (230.7)^2} \\
&= 461.4 \text{ volts} \\
\tan \theta_0 &= \frac{230.7}{399.6} = 0.578 \\
\theta_0 &= 30 \text{ degrees}
\end{aligned}$$

The magnitude of the voltage $\bar{E}_0 = \bar{E}_1 - \bar{E}_2$ is, therefore, equal to 461.4 volts and leads the voltage E_1 by 30 degrees. It leads the voltage E_2 by $120 + 30 = 150$ degrees. The expression for its instantaneous value is

$$\begin{aligned}
e_0 &= \sqrt{2} \times 461.4 \sin (2\pi 60t + 30°) \\
&= 652.4 \sin (377t + 30°)
\end{aligned}$$

Since the circuit contains nothing but resistance, the voltage impressed across it must be equal to the resistance drop at each instant. Since the resistance is constant, the current i through the resistance must be proportional to the voltage at each instant. Hence

$$i = \frac{\sqrt{2}E_0 \sin (\omega t + 30°)}{r}$$

$$\begin{aligned}
I_{r.m.s.} = \frac{E_{r.m.s}}{r} &= \frac{461.4}{10} \\
&= 46.14 \text{ amperes} \\
I_m &= \sqrt{2} \times 46.14. \\
&= 65.24 \text{ amperes}
\end{aligned}$$

Since the current is in phase with the voltage, the expression for the instantaneous current is

$$
\begin{aligned}
i &= \sqrt{2} \times 46.14 \sin (377\, t + 30^\circ \pm 0^\circ) \\
&= 65.24 \sin (377t + 30^\circ)
\end{aligned}
$$

The complex expression for the current in terms of its root-mean-square value is

$$
\begin{aligned}
\bar{I} = \frac{\bar{E}_0}{r} &= \frac{399.6 + j230.7}{10} \\
&= 39.96 + j23.07
\end{aligned}
$$

The vector expression for the current may also be found by making use of the operator which rotates a vector through a given angle. Since the current is in phase with the voltage, it must be displaced from the axis of reals by the same angle as the voltage E_0 producing it, *i.e.*, by an angle of 30 degrees. The operator which displaces a vector to which it is applied by an angle θ is $(\cos \theta + j \sin \theta)$. (See Chapter I, page 7.) Therefore,

$$
\begin{aligned}
\bar{I} &= I (\cos \theta + j \sin \theta) \\
&= 46.14 (\cos 30^\circ + j \sin 30^\circ) \\
&= 46.14 (0.866 + j0.500) \\
&= 39.96 + j\, 23.07
\end{aligned}
$$

From the definition of the effective or root-mean-square value of a current it follows that the average power absorbed in the resistance is

$$
\begin{aligned}
P &= I^2 r \\
&= (46.14)^2 \times 10 \\
&= 21{,}290 \text{ watts}
\end{aligned}
$$

CHAPTER III

POWER WHEN CURRENT AND VOLTAGE ARE SINUSOIDAL

Power Absorbed and Delivered by a Circuit.—The problems concerning power which arise in dealing with direct-current circuits are usually simple, and there is seldom any doubt as to whether power is absorbed or delivered. The conditions are not always so simple with alternating currents. It is necessary, therefore, to determine definitely the conditions under which power is absorbed and delivered. The fundamental conditions are the same for direct-current and alternating-current circuits.

When a direct-current dynamo acts as a generator, the current flow is from the plus to the minus terminal through the external circuit and it is from the minus to the plus terminal through the armature of the dynamo. The machine is delivering power. The load is absorbing power. Through the load the current flow is from a higher to a lower potential or in the direction of decreasing potential, *i.e.*, it is in the direction of the potential drop. Through the dynamo the current flow is from a lower to a higher potential or in the direction of increasing potential, *i.e.*, it is in the direction of the potential rise. When the dynamo acts as a motor, the current flows through the armature from the plus to the minus terminal or in the direction of the potential drop. In this case power is being absorbed by the dynamo. In general, if there is a rise in potential in the direction of the current flow, power is being generated and power is delivered. If there is a drop or fall in potential in the direction of current flow, power is absorbed.

The conditions under which power is absorbed or power delivered by an alternating-current circuit are exactly the same as those for a direct-current circuit. Due, however, to the difference in phase of the current and voltage in most alternating-current circuits, the current is seldom in the direction of either the

voltage rise or the voltage fall during all parts of a cycle. Consequently, power is absorbed during part of each cycle and delivered during the remaining part. If the average power absorbed during a cycle is greater than the average power delivered, the net effect is power absorbed. If the average power delivered during a cycle is greater than the average power absorbed, the net effect is power delivered.

Instantaneous Power.—The instantaneous power p in a circuit is given by the product of the instantaneous values of the voltage and current:

$$p = ei$$

If e is a voltage rise in the direction of current flow, p represents power delivered. When e is a fall in voltage in the direction of current flow, power is absorbed. Whether power delivered or power absorbed is considered positive or negative depends upon the convention adopted for the positive direction of a voltage rise.

If the voltage e is considered positive when it represents an actual rise in voltage in the direction assumed positive for current flow, the expression $p = ei$ represents power delivered. When the product of e and i is positive, power is actually delivered, since the voltage is then actually increasing in the direction of current flow. When the product of e and i is negative, the voltage is actually decreasing in the direction of current flow and power is absorbed. In the latter case there is a fall of potential, *i.e.*, a negative rise in potential, in the direction of current flow and the power delivered is negative. Negative power delivered is power absorbed. According to this convention, a voltage rise in the positive direction of current flow is positive. The negative of a voltage rise is a voltage fall.

If the positive direction for a voltage rise is taken opposite to the positive direction for current flow, the expression $p = ei$ represents power absorbed. Power is actually absorbed when $p = ei$ is positive, and it is actually delivered when $p = ei$ is negative. According to this convention, a voltage drop, or fall in potential in the positive direction of current flow, is considered positive.

Curves of e, i and $p = ei$ are plotted in Fig. 16. If e represents a voltage rise, considered positive in the positive direction of current flow, power is delivered when e and i have the same sign, *i.e.*, when both are positive or both are negative. There is then an actual rise in voltage in the direction of current flow. When e and i have opposite signs, power is absorbed. In this case, there is an actual fall in voltage in the direction of current flow. Power delivered is positive and power absorbed is negative.

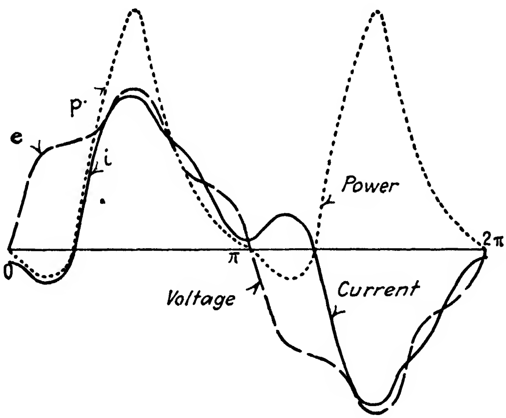


FIG. 16.

When only power delivered is to be considered, it is better to take the positive direction for a voltage rise in the direction assumed positive for current flow, since this convention makes power delivered positive and avoids the use of negative signs before the expressions for power delivered. This convention is better, and also more logical, when power delivered and power absorbed are involved in the same problem. When, however, only power absorbed is to be considered, it is better to adopt the convention that voltage drops are positive in the direction assumed positive for current flow, since then power absorbed is positive and the necessity of using minus signs before the expressions for power absorbed is avoided.

Average Power.—The power delivered or absorbed by an alternating-current circuit is the average power considered over

a complete cycle. It is seldom that the instantaneous power is of interest. The average power is given by

$$P = \frac{1}{T}\int_0^T eidt \tag{1}$$

This expression gives the mean ordinate of the power curve, *i.e.*, the dotted curve in Fig. 16, and is equal to the net area enclosed by the power curve divided by its base. In order to evaluate the integral, it is necessary to know the forms of the functions determining the voltage e and the current i.

Power when the Voltage and Current Waves Are Both Sinusoidal.—Assume the voltage and current waves to be sinusoidal.

CASE I. VOLTAGE AND CURRENT IN PHASE.—Let

$$e = E_m \sin \omega t$$
$$i = I_m \sin \omega t$$
$$p = ei = E_m I_m \sin^2 \omega t = E_m I_m \frac{1 - \cos 2\omega t}{2} \tag{2}$$

$$\text{Average power} = P = \frac{1}{T}\int_0^T \frac{E_m I_m}{2}\, dt - \frac{1}{T}\int_0^T \frac{E_m I_m}{2} \cos 2\omega t\, dt$$
$$= \frac{E_m I_m}{2} = \frac{E_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$
$$= EI \tag{3}$$

where E and I without subscripts are the root-mean-square or effective values of the voltage and current. Therefore, when the voltage and current are in phase, the average power is equal to the product of the effective values of voltage and current. It is equal to the effective volt-amperes, *i.e.*, the product of the effective voltage and effective current.

The expression for the instantaneous power p, equation (2), may be written

$$p = P - \frac{E_m I_m}{2} \cos 2\omega t = P - P \cos 2\omega t \tag{4}$$

This equation and the equations for voltage and current are plotted in Fig. 17. It should be noted that the power curve is sinusoidal in form and has double frequency as compared with the voltage and current. Its axis of symmetry is at a distance P, equal to the average power, above the axis of the voltage and

current curves. Its amplitude is equal to one-half the maximum volt-amperes or is equal to the effective volt-amperes. Effective volt-amperes are equal to average power when the voltage and current are in phase. Although the power is not constant, its flow is always in the same direction when the voltage and current are in phase. The corresponding instantaneous values of voltage and current are always of the same sign. The current flow is, therefore, always either in the direction of the voltage rise or in the direction of the voltage drop, according as *e*, in the equation for voltage, represents a voltage rise or a voltage drop. If *e* represents a voltage rise, the expression for average power gives

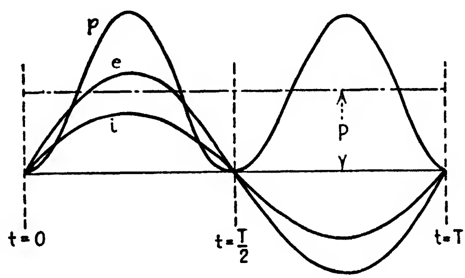


FIG. 17.

the power delivered. If *e* represents a voltage drop, the expression for average power gives the power absorbed. If *e*, representing a voltage rise, and *i* are opposite in phase, the expression for power is negative and represents power absorbed. If *e*, representing a voltage drop, and *i* are opposite in phase, the expression for power represents power delivered.

CASE II. VOLTAGE AND CURRENT IN QUADRATURE.—Let

$$\begin{aligned} e &= E_m \sin \omega t \\ i &= I_m \sin (\omega t - 90°) \\ p = ei &= E_m I_m (\sin^2 \omega t \cos 90° - \sin \omega t \cos \omega t \sin 90°) \\ &= E_m I_m (0 - \sin \omega t \cos \omega t) \\ &= \frac{E_m I_m}{2} (0 - \sin 2\omega t) \qquad (5) \end{aligned}$$

$$\text{Average power} = P = \frac{1}{T}\int_0^T \frac{E_m I_m}{2} (0 - \sin 2\omega t)dt$$
$$= 0$$

When the voltage and current are in quadrature, the average power is zero.

The curves for instantaneous power, equation (5), and for the instantaneous voltage and current are plotted in Fig. 18. The power curve is again a simple harmonic curve of double frequency with respect to the voltage and current and it still has an amplitude equal to the effective volt-amperes, but its axis of symmetry now coincides with the axis of symmetry of the voltage and current. This latter condition follows from the fact that the average power P is zero.

Although the average power is zero, the power at any instant is not zero except at four points during each cycle. There is an

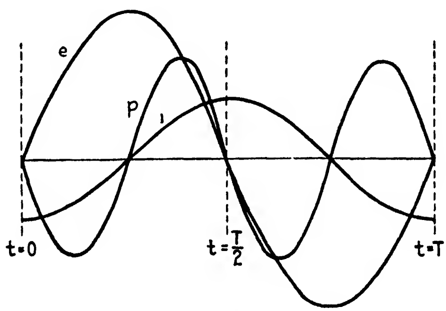


FIG 18.

oscillation of power between the source and the load, the average value of which is zero. The amplitude of this oscillation is equal to EI, the effective volt-amperes of the circuit. If e represents a voltage rise, the positive power loops represent power delivered. The negative power loops represent power absorbed. During two quarters of each cycle, power is delivered by the circuit. During the other two quarters of each cycle, an equal amount of power is absorbed. The power which is absorbed is stored as kinetic energy either in the rotating part of a motor or generator or in a magnetic field produced by the current, or it may be stored as potential energy in the electrostatic field of a condenser. This stored kinetic or potential energy is given back to the circuit as dynamic electrical energy.

CASE III. GENERAL CASE. VOLTAGE AND CURRENT NEITHER IN PHASE NOR IN QUADRATURE.—Let

$$e = E_m \sin \omega t$$
$$i = I_m \sin (\omega t - \theta)$$

where θ is neither 0 nor 90 degrees.

$$p = ei = E_m I_m \sin \omega t \sin (\omega t - \theta) \tag{6}$$

Since $2 \sin \alpha \sin \beta = \cos (\alpha - \beta) - \cos (\alpha + \beta)$, equation (6) may be written

$$p = \frac{E_m I_m}{2}\{\cos (+\theta) - \cos (2\omega t - \theta)\} \tag{7}$$

$$\text{Average power} = P = \frac{1}{T}\int_0^T \frac{E_m I_m}{2}\{\cos (+\theta) - \cos (2\omega t - \theta)\}\, dt$$
$$= \frac{E_m I_m}{2} \cos (+\theta) + 0 = \frac{E_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}} \cos \theta$$
$$= EI \cos \theta \tag{8}$$

The average power is equal to the product of the effective values of voltage and current multiplied by the cosine of the phase angle between the voltage and current. When the phase angle is zero, $P = EI \cos \theta$ becomes EI. When the phase angle is 90 degrees, $P = EI \cos \theta$ becomes zero.

Equation (7) for the instantaneous power may be written

$$p = EI \cos \theta - EI \cos (2\omega t - \theta)$$
$$= P - EI \cos (2\omega t - \theta) \tag{9}$$

From equation (9) it may be seen that, in the general case as well as in the two special cases first considered, the power curve is a double-frequency curve. As in the first two cases it has an amplitude equal to the effective volt-amperes and its axis is displaced from the axis of voltage and current by a distance equal to the average power. Curves of power, voltage and current are plotted in Fig. 19.

If e is taken as a voltage rise, equation (8) represents power delivered. Power is actually delivered when $EI \cos \theta$ is positive. Power is actually absorbed when $EI \cos \theta$ is negative. It is positive when θ is less than 90 degrees, since the cosine of an angle which is less than 90 degrees is positive. It is negative when θ is greater than 90 degrees (but less than 270 degrees), since the cosine of an angle which is greater than 90

degrees (but less than 270 degrees) is negative. Since an angle of lag of θ degrees is the same as an angle of lead of 360 minus θ degrees and similarly an angle of lead of θ degrees is the same as an angle of lag of 360 minus θ degrees, it is customary to use the smaller of the two angles when expressing the phase difference of a current and a voltage. Equal angles of lead and lag produce the same effect so far as average power is concerned.

When $\theta = 0$, the power curve, Fig. 19, is entirely above the axis of current and voltage. This corresponds to the conditions shown in Fig. 17, page 56, for Case I where the voltage and current are in phase. When θ is greater than zero and less than 90 degrees, the axis of the power curve is displaced in a positive

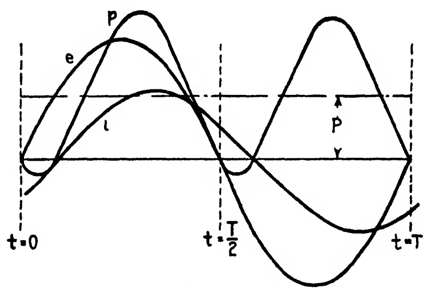


FIG. 19.

direction from the axis of the voltage and current curves. In this case (assuming E represents a voltage rise), power is delivered. If θ is greater than 90 degrees, the axis of the power curve is displaced in a negative direction from the axis of the voltage and current curves. In this case, the average power is negative and power is absorbed. If θ is equal to 90 degrees, power is neither delivered nor absorbed. This corresponds to Case II.

Volt-amperes; Apparent or Virtual Power.—The volt-amperes, or apparent or virtual power of a circuit, is equal to the product of the effective or root-mean-square values of voltage and current. This product is not equal to the true power except when the voltage and current are in phase. Although volt-amperes do not represent true power except in the case mentioned, a knowledge of the volt-amperes of a circuit is usually of considerable importance, since volt-amperes and not watts determine the limit of output of most electrical apparatus. The limit of output

of all alternating-current apparatus, such as motors, generators, transformers etc., is determined chiefly by the rise in temperature produced in the windings. The increase in temperature is caused principally by the core and copper losses. Core losses depend upon frequency and flux density and are fixed by the operating voltage and frequency. Copper losses are determined by the current. For fixed voltage and current, these losses are practically independent of the phase relation between current and voltage. The limit of output of a generator, motor or transformer, therefore, is determined by the volt-amperes which produce the limiting rise in temperature in the windings. All types of alternating-current apparatus, which can operate with different phase angles between current and voltage, *i.e.*, at different power factors, are rated in volt-amperes. Full load is reached when they carry full-rated current at rated voltage and frequency. Their output under this condition is zero if the current and voltage are out of phase by 90 degrees. It is a maximum when the current and voltage are in phase. This latter condition is most to be desired and is approached in practice, although seldom exactly attained. The common unit for apparent power for large power apparatus is the kilovolt-ampere (abbreviated kva.). This bears the same relation to the volt-ampere as the kilowatt bears to the watt.

Power Factor.—For steady voltages and currents, the power is always given by the product of volts and amperes, *i.e.*, by the volt-amperes. For alternating currents, this is true only when the voltage and the current are in phase. When the voltage and the current are not in phase, the product of the effective voltage and the effective current must be multiplied by a factor, known as the *power factor*, in order to get the actual power. The power factor is always equal to the ratio of the average power to the product of the effective voltage and the effective current, *i.e.*,

$$\text{Power factor} = \frac{P}{VI}$$

For sinusoidal waves, power is given by $EI \cos \theta$, equation (8). In this case the power factor is $\cos \theta$, *i.e.*, it is the cosine of the phase angle between the voltage and the current. When the

voltage and the current are not sinusoidal, the factor $\cos\theta$ has no significance except when considered with respect to the equivalent sine waves. These are explained later.

Theoretically, the power factor may have any value from zero to unity, both inclusive. Although it is possible to obtain zero power factor experimentally by the use of a synchronous motor which receives enough power mechanically to supply its losses, zero power factor is not reached in practice. It would be a very undesirable condition in most cases. Unity power factor may be obtained and is not uncommon in alternating-current work.

The power factor of a circuit is always the ratio of the true power to the apparent power, *i.e.*, the ratio of the watts to the volt-amperes. It is merely a ratio and, therefore, independent of the units used except that the true power and the apparent power must be expressed in the same system of units. Power factor is commonly expressed in per cent, although it is frequently given as a decimal. It must always be taken as a decimal when used in numerical expressions. Power factor is the cosine of an angle only for sinusoidal waves of current and voltage.

$$\text{Power factor} = \frac{\text{watts}}{\text{volt-amperes}} \tag{10}$$

This expression for power factor holds regardless of wave form.

Reactive Factor.—The expression

$$\begin{aligned}\sin\theta &= \sqrt{1 - (\cos\theta)^2} \\ &= \sin\left[\cos^{-1}(\text{power factor})\right]\end{aligned} \tag{11}$$

is called the *reactive factor*. It is true, however, only for sinusoidal waves. When the waves are sinusoidal, the reactive factor is also given by

$$\text{Reactive factor} = \sqrt{1 - (\text{power factor})^2} \tag{12}$$

The reactive factor is of considerable importance, since a knowledge of its magnitude is necessary when determining the size of the apparatus required to raise the power factor of a circuit to unity, or by any desired amount.

Measurement of Average Power.—The average power in a circuit may be measured by an electrodynamometer. If the fixed

coil of such an instrument is placed in series with one main of the circuit in which the power is to be measured, and its movable coil is connected in series with a large non-inductive resistance and then shunted between the two mains, the average torque developed between the two coils is proportional to the average product of the instantaneous values of current and voltage. This is because the torque exerted on two coils, one of which is free to turn about a common diameter, is proportional to the product of the currents they carry. In the electrodynamometer used as a wattmeter, the fixed coil (usually called the *current* coil) carries line current. The movable coil (usually called the *potential* coil), in series with non-inductive resistance, carries a current which is proportional to the voltage across its terminals, provided the non-inductive resistance is large compared with the inductance of the potential windings of the instrument.

$$\begin{aligned}\text{Average torque} &= k\frac{1}{T}\int_0^T i_p i_c \, dt \\ &= k\frac{1}{T}\int_0^T \frac{e_p}{R} i_c \, dt \\ &= k\frac{1}{R} P\end{aligned}$$

where i_p and i_c are, respectively, the instantaneous currents carried by the potential and current coils, and k is a constant of proportionality. R is the non-inductive resistance in series with the potential coil plus the resistance of the potential coil and e_p is the instantaneous voltage across the terminals of the potential coil circuit P is the average power. The movable coil deflects until the average torque developed in it is just balanced by the torsion of the control springs attached to its shaft. By proper calibration the instrument may be made to indicate the average power. When the power in high-voltage circuits is to be measured, it is necessary to use transformers with both the current and the potential coils of the instrument.

Measurement of Power Factor.—Although there are special instruments which indicate the power factor of the circuit in which they are connected, they are not very accurate and are not in general use except on switchboards. The power factor

of a circuit is generally determined by measuring the voltage, current and power by means of a voltmeter, an ammeter and a wattmeter, and then taking the ratio of the watts to the volt-amperes. The reactive factor is calculated from the power factor by equation (12), page 61.

Active and Reactive or Quadrature Components of Current.—When dealing with sinusoidal waves, it is often convenient to resolve the current into two right-angle components, one in phase with the voltage, the other in quadrature with it. These two components are $I \cos \theta$ and $I \sin \theta$. The reason for selecting these two components will be made clear by what follows. Let

$$e = E_m \sin \omega t$$
$$i = I_m \sin (\omega t - \theta)$$

Then, since $i = I_m \cos \theta \sin \omega t - I_m \sin \theta \cos \omega t$,

$$p = E_m \sin \omega t\,(I_m \cos \theta) \sin \omega t - E_m \sin \omega t\,(I_m \sin \theta) \cos \omega t \quad (13)$$

$$P = \frac{1}{T}\int_0^T E_m \sin \omega t\,(I_m \cos \theta) \sin \omega t\,dt - \frac{1}{T}\int_0^T E_m \sin \omega t\,(I_m \sin \theta) \cos \omega t\,dt \quad (14)$$

The second integral vanishes and, therefore, contributes nothing to the average power or watts of the circuit. The power may be regarded as due to a voltage $e = E_m \sin \omega t$ and a current $i = (I_m \cos \theta) \sin \omega t$, whose maximum values are E_m and $I_m \cos \theta$, respectively, and whose phase difference is zero. Since the power is contributed wholly by the component of the current $I_m \cos \theta$, which is in phase with E_m, this component is known as the *active*, *power* or *energy component* of the current, or simply the *active*, *power* or *energy* current. The same terms are applied to the corresponding component of the effective current. In fact, when active, power or energy current is mentioned, the active, power or energy component of the effective current is understood.

The second integral may be regarded as representing the power due to a voltage $e = E_m \sin \omega t$ and a current $i = -(I_m \sin \theta) \cos \omega t$ or $i = (I_m \sin \theta) \sin (\omega t - 90°)$, whose maximum values are respectively E_m and $I_m \sin \theta$ and whose phase difference is 90

degrees. It has already been shown that when a voltage and a current are in quadrature, the average power is zero. The component $I_m \sin\theta$ of the current contributes nothing to the average power of the circuit and is, therefore, known as the *reactive, wattless* or *quadrature* component of the current, or simply the *reactive, wattless* or *quadrature* current. The same terms are applied to the corresponding component of the effective current. Although the component $I_m \sin\theta$ contributes nothing to the average power of the circuit, its presence increases the resultant current and, therefore, increases the copper loss for a given amount of power. The resultant current is always equal to the square root of the sum of the squares of the active and the reactive components of the current. The resultant current causes the copper loss but only its active component contributes to the average power.

If E and I represent the root-mean-square values of the voltage and current, the expression

$$P_a = EI \cos\theta = EI \times \text{power factor}$$

represents the average power due to the active component of the current. P_a is the true average power of the circuit. It is sometimes called the *active* power or the *active* volt-amperes, *i.e.*, the volt-amperes due to the *active* component of the current.

The second term of the second member in equation (14) is the average power due to the reactive component of the current. This average is zero. The expression for the *instantaneous* power due to the reactive component of the current is

$$\begin{aligned} p_r &= -E_m I_m \sin\theta \sin\omega t \cos\omega t \\ &= -\frac{E_m I_m}{2} \sin\theta \sin 2\omega t \end{aligned} \tag{15}$$

This has double frequency and represents an oscillation of power between the generating source and the load, of maximum value

$$\frac{E_m I_m}{2} \sin\theta = EI \sin\theta \tag{16}$$

In addition to representing the maximum value of the power oscillation caused by the reactive component of the current

equation (16), *i.e.*, $\frac{E_m I_m}{2} \sin\theta = EI \sin\theta$, also represents the root-mean-square volt-amperes due to the *reactive* component of the current. It is called the *reactive* or *quadrature* power. A better name is *reactive* volt-amperes, *i.e.*, the volt-amperes due to the *reactive* or *quadrature* component of the current. Reactive power or reactive volt-amperes is the product of the total volt-amperes and the reactive factor.

The unit for reactive power which was adopted by the International Electrotechnical Commission in Stockholm in 1930 is the *var*. The word var is made of the initial letters of volt-ampere reactive. A var is the reactive power corresponding to one reactive volt-ampere, *i.e.*, it is equal to the product of one volt and one reactive ampere. The kilovar is one thousand vars.

The total volt-amperes of a circuit are equal to the square root of the sum of the squares of the active and reactive powers.

$$\begin{aligned}
\text{Volt-amperes} &= \sqrt{(EI\cos\theta)^2 + (EI\sin\theta)^2} \\
&= \sqrt{(\text{active power})^2 + (\text{reactive power})^2} \qquad (17) \\
&= \sqrt{(\text{watts})^2 + (\text{vars})^2} \\
&= EI
\end{aligned}$$

It follows from equation (17) that with sinusoidal waves, power factor may be defined as the ratio of the active power to the square root of the sum of the squares of the active and reactive powers.

Although the average value of the oscillation of power caused by the reactive component of the current is zero, the oscillation cannot take place without producing a copper loss. The oscillation cannot contribute to the available power, but it does increase the copper loss in a circuit and also increases the size of the apparatus necessary to supply the power. For this reason it is desirable to operate circuits at as nearly unity power factor as possible.

The active and reactive components of current for a circuit carrying I amperes at a power factor $\cos\theta$ are

$$\text{Active current} = I\cos\theta = I \times \text{power factor} \qquad (18)$$

$$\begin{aligned}
\text{Reactive current} = I\sin\theta &= I\sqrt{1-\cos^2\theta} \\
&= I\sqrt{1-(\text{power factor})^2} \\
&= I \times \text{reactive factor} \qquad (19)
\end{aligned}$$

For sinusoidal waves, the watts are always given by the product of volts and active amperes. Vars are always given by the product of volts and reactive amperes.

Curves of voltage, active and reactive components of the current and the power curves corresponding to the two components of the current are shown in Fig. 20. On this figure the subscripts a and r indicate the active and reactive components, respectively. The two power curves are obtained by plotting the two terms of the second member of equation (14) separately against time. The two component currents $I_m \cos \theta$ and $I_m \sin \theta$ are assumed to be equal. This corresponds to a power factor

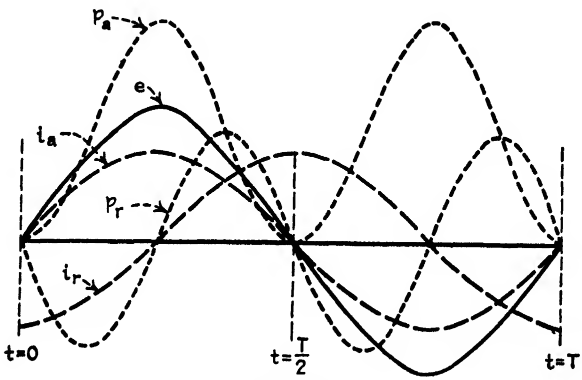


FIG 20

of 0.707 and a phase angle of 45 degrees between voltage and current. The phase angle is assumed to be an angle of lag of the current with respect to the voltage.

The power curves corresponding to both components of the current are double-frequency curves and represent a periodic flow of power. The power due to the active component of the current, $I_m \cos \theta$, is always positive and represents the power given out by the circuit. The power due to the reactive component of the current, $I_m \sin \theta$, alternates between plus and minus, and the average power is zero. It represents an oscillation of energy in the circuit the average value of which is zero. The combined effect of the two components produces a power curve partly above and partly below the axis of time, similar to that shown in Fig. 19, page 59.

At certain instants the resultant power developed is positive and at others it is negative. (See Fig. 19.) The average power during any cycle is found by subtracting the area enclosed between the axis of time and the negative loops from the area enclosed between this axis and the positive loops, and dividing the remainder by $T = \frac{1}{f}$. The resultant area is the same as the area enclosed between this axis and the power curve corresponding to the energy component of the current, $I_m \cos \theta$.

The conditions shown in Figs. 17 and 18 represent limiting cases. In the first, the reactive component of the current is zero. In the second, the active component is zero.

It is important to remark that the power in a single-phase circuit is always fluctuating, becoming zero or even negative at certain instants of time. This means that a circuit may supply power to the generator during part of a cycle. The conditions are different in a balanced polyphase circuit. Here the deficiency of power at any instant in one phase is made up by an excess of power in the other phases. The *total* power of such a circuit, *i.e.*, the resultant power for all the phases, does not fluctuate.

Active and Reactive or Quadrature Components of the Voltage. Instead of dividing the current into active and reactive components with respect to the voltage, the voltage may be similarly divided into active and reactive components with respect to the current.

The two components of the voltage are

$$e_a = (E_m \cos \theta) \sin \omega t \tag{20}$$

$$e_r = -(E_m \sin \theta) \cos \omega t = (E_m \sin \theta) \sin (\omega t - 90°) \tag{21}$$

In this case the power may be considered to be due to a voltage $e = (E_m \cos \theta) \sin \omega t$ and a current $i = I_m \sin \omega t$ in phase with it. The component $e = -(E_m \sin \theta) \cos \omega t$ of the voltage is in quadrature with the current $i = I_m \sin \omega t$ and, therefore, produces no average power. The component $E_m \cos \theta$ is known as the *active, power* or *energy* component of the voltage. The component $E_m \sin \theta$ is known as the *reactive, wattless* or *quadrature* component of the voltage.

Measurement of Reactive Power, i.e., Reactive Volt-amperes or Vars.—The reactive power of a circuit may be measured by an

electrodynamometer connected in the same way as for measuring true power, provided the current through its potential circuit can be made to lag the voltage across its terminals by 90 degrees. For measurement of true power, the current in the potential circuit may be in phase with the voltage across its terminals. The necessary lag of 90 degrees for the current in the potential coil may be obtained by connecting it in series with a large inductance instead of a large non-inductive resistance as is done for power measurements. (See page 61.) Due to the fact that an inductance cannot be made without some resistance, the current in the potential coil does not lag by exactly 90 degrees if the series inductance alone is used. It may, however, be made to lag exactly 90 degrees by shunting the potential coil with a suitable non-inductive resistance. Since the effect of inductance depends upon frequency, a reactive-power meter made in this way indicates $EI \sin \theta$ only when the frequency is that for which the instrument is adjusted. It cannot be used with non-sinusoidal waves. The effect of the inductance and shunted resistance will be understood after studying Chapter VII.

The average torque developed on the coils of the electrodynamometer is

$$\text{Average torque} = k\frac{1}{T}\int_0^T i_p i_c dt$$

$$= k\frac{1}{T}\int_0^T KE_m \sin(\omega t - 90°) I_m \sin(\omega t - \theta) dt$$

where E_m and I_m are the maximum values of the voltage and current of the circuit and K is a constant of proportionality between the voltage $E_m \sin \omega t$ across the potential circuit and its current i_p. This assumes a pure inductive circuit.

$$\text{Average torque} = kK\frac{E_m I_m}{2}\sin\theta$$

$$= kKEI \sin\theta$$

$$= kK \times \text{reactive power}$$

By suitable calibration, the instrument may be made to indicate reactive power directly.

✓ **Vectors Representing Voltage Rise and Voltage Fall.**—When plotting vector diagrams and in making computations involving

currents and voltages, it is always necessary to distinguish between voltage rise and voltage fall or drop.

If the direction *ab* in any circuit is taken positive for current flow, the current is actually positive and actually flows from *a* to *b* when the revolving vector representing the current I_{ab} lies in the first or second quadrant. It is actually negative and flows from *b* to *a* when the revolving vector representing it lies in the third or fourth quadrant. If *ab* is the positive direction for the current, it is also the positive direction for the voltage V_{ab} associated with the current. If the vector V_{ab} represents a voltage rise, then there is an actual voltage rise from *a* to *b*, *i.e.*, *b* is at a higher potential than *a* when the voltage vector V_{ab} lies in the first or second quadrant. When it lies in the third or fourth quadrant, its projection on the vertical axis is negative and there is a negative rise in potential from *a* to *b* or an actual fall in potential from *a* to *b*, *i.e.*, *b* is at a lower potential than *a*. Under these conditions there is an actual rise in potential from *b* to *a*. If on the average during a cycle there is a voltage rise in the direction of current flow, then, on the average, power is delivered. Power is delivered when the phase angle between the vector representing voltage rise and the current vector is less than 90 degrees, *i.e.*, when the active component of the voltage rise is in phase with the current; in other words, when it is positive with respect to the current. Under this condition, the component of voltage in phase with the current, *i.e.*, its active component, is a rise in the positive direction *ab* when the current flow is positive. It is still a voltage rise in the direction of current flow when its sign has reversed. It is rising in a negative direction with respect to the direction assumed positive for current flow, but the direction of current flow has also reversed.

If the phase between the vector V_{ab}, representing voltage rise, and the current vector I_{ab} is greater than 90 degrees, *i.e.*, if the active component of voltage rise with respect to current is negative, then on the average during a cycle there is a voltage fall in the direction of current flow. The average product of e and i is negative and power is absorbed. When voltage rise is assumed positive, power delivered is positive and power absorbed is negative.

If ab is assumed to be the positive direction of voltage drop, the vector V_{ab} represents a voltage drop. Under this condition, if the angle between the active component of voltage drop and current is less than 90 degrees, then, on the average, during a cycle there is a voltage drop in the direction of current flow and, on the average, power is absorbed. When the angle is greater than 90 degrees, the active component of voltage drop with respect to current is negative, and, on the average, there is a voltage rise in the direction of current flow and power is delivered. When voltage drops are assumed positive, power absorbed is positive and power delivered is negative.

When a voltage rise is considered positive, power delivered is positive and power absorbed is negative. When a voltage drop is considered positive, power absorbed is positive and power delivered is negative.

In general, when only power delivered or both power delivered and absorbed are to be considered in the same problem, it is best to let a positive vector represent a voltage rise. A negative vector then represents a voltage fall or drop. According to this convention, a vector V represents a voltage rise in a circuit in the direction which is assumed positive for the current. A vector $-V$ is the voltage drop in the same direction or a rise in the opposite direction. However, as has been stated, when dealing only with power absorbed it is convenient and customary to consider that a positive vector represents a voltage drop. When this latter convention is adopted, V represents a voltage drop in the direction in which the current is considered positive and $-V$ represents a voltage rise in the same direction.

Expression for Power when the Voltage and Current Are in Complex.—The average or active power in a circuit is not a vector, and it cannot be found from the product of the vectors representing the current and voltage. It is the scalar product of the effective voltage, the effective current and the cosine of the angle between the vectors representing the voltage and current. Let

$$\bar{E} = e + je'$$
$$\bar{I} = i + ji'$$

where E and I are the effective or root-mean-square values of the voltage and current. The small letters without primes are the

real components; with primes they represent the j components. The two vectors $\bar{E}$ and $\bar{I}$ are shown in Fig. 21.

From Fig. 21

$$
\begin{aligned}
P = P_a &= EI \cos \theta \\
&= EI \cos (\theta_1 - \theta_2) \\
&= EI(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\
&= EI\left(\frac{e}{E} \times \frac{i}{I} + \frac{e'}{E} \times \frac{i'}{I}\right) \\
&= ei + e'i' \qquad (22)
\end{aligned}
$$

The component e of the voltage cannot produce power with the component i' of the current, since e and i' are in phase quadrature. Neither can the component e' of the voltage produce power with the component i of the current, since they are also in phase quadrature. The components in the products ei and $e'i'$ are in phase. They are the only components that can contribute to the power.

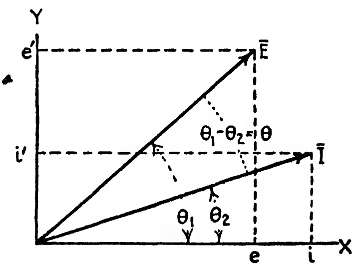


FIG. 21.

In general, when finding average power, the voltage and current may each be divided into any number of components. When so divided, the average power is equal to the algebraic sum of the component powers due to each component of voltage considered with respect to each component of current. The voltage and current in Fig. 21 are each divided into two components. In this case, the power is

$$P = ei \cos \theta^e_i + ei' \cos \theta^e_{i'} + e'i \cos \theta^{e'}_i + e'i' \cos \theta^{e'}_{i'} \qquad (23)$$

For the components chosen, the angles between two of the pairs of components are zero, and between the other two pairs of components they are 90 degrees. Substituting the angles in equation (23) gives

$$
\begin{aligned}
P &= ei \cos 0° + ei' \cos 90° + e'i \cos 90° + e'i' \cos 0° \\
&= ei + 0 + 0 + e'i' \qquad (24)
\end{aligned}
$$

which is the same as found in equation (22).

Expression for Reactive Power when the Current and the Voltage Are in Complex.—Refer to Fig. 21.

$$\begin{aligned}\text{Reactive power} = P_r &= EI \sin \theta \\ &= EI \sin (\theta_1 - \theta_2) \\ &= EI(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2) \\ &= EI\left(\frac{e'}{E} \times \frac{i}{I} - \frac{e}{E} \times \frac{i'}{I}\right) \\ &= e'i - ei' \qquad (25)\end{aligned}$$

This same expression may be obtained by taking the algebraic sum of the reactive powers due to each component of voltage with respect to each component of current. In deriving equations (22) and (25), the angle $(\theta_1 - \theta_2)$ between the voltage and current was used. This is the angle of lead of the voltage with respect to the current. The angles of lead of the voltage with respect to the current will be used in the following equations.

$$\begin{aligned}P_r &= ei \sin \theta_i^e + ei' \sin \theta_{i'}^e + e'i \sin \theta_i^{e'} + e'i' \sin \theta_{i'}^{e'} \\ &= ei \sin (0°) + ei' \sin (-90°) + e'i \sin (90°) + e'i' \sin (0°) \\ &= 0 - ei' + e'i + 0 \qquad (26)\end{aligned}$$

It should be noted that the power $P = ei + e'i'$ is not given by the product of the complex values of current and voltage. This product is evidently $(ei - e'i') + j(ei' + e'i)$ and has no significance.

If the angles had been taken as the angles of lead of current with respect to the voltage, the signs of the terms appearing in the equation for the reactive power, and therefore the sign of the reactive power, would have been reversed. No effect would have been produced in the sign of the active power since the algebraic signs of the cosines of equal positive and negative angles are the same. The sign of reactive power depends not only on whether the current leads or lags the voltage but also on whether the angles are taken as angles of lead of the voltage with respect to the current or angles of lead of the current with respect to the voltage.

When θ, the phase angle between voltage and current, is taken as the angle of the voltage with respect to the current, reactive power is positive for lagging current and negative for leading current. When θ is taken as the angle of the current

with respect to the voltage, reactive power is negative for lagging current and positive for leading current. The Committee on Electrical and Magnetic Units, at a meeting in Paris in 1934, adopted the convention that reactive power for a leading current shall be positive.

Active and Reactive Powers by Using the Conjugate of either Voltage or Current.—The active and reactive powers may be found from the complex expressions of voltage and current, either by multiplying the complex expression for the conjugate of the current with respect to the axis of reference by the complex expression for the voltage or by multiplying the complex expression for the conjugate of the voltage with respect to the axis of reference by the complex expression for the current. The sign of the active power is the same in both cases, but the sign of the reactive power is opposite in the two cases. The sum of the real terms in the complex product is the active power. The sum of the j terms is the reactive power. For example, referring to Fig. 21, the vector expressions for the current and the voltage are $\bar{E} = e + je'$ and $\bar{I} = i + ji'$. The conjugate of the current with respect to the axis of reference is $\bar{I}_c = i - ji'$.

$$\begin{aligned} \bar{E} \times \bar{I}_c &= (e + je')(i - ji') \\ &= (ei + e'i') + j(e'i - ei') \\ &= P + jQ \end{aligned} \tag{27}$$

The real part of this expression, *i.e.*, $(ei + e'i')$ is the active power. [See equation (22).] The imaginary or j part, *i.e.*, $(e'i - ei')$, is the reactive power. [See equation (25).]

$$\begin{aligned} EI &= \sqrt{(P)^2 + (Q)^2} \qquad (28) \\ &= \sqrt{(\text{active power})^2 + (\text{reactive power})^2} \\ &= \sqrt{(\text{active volt-amperes})^2 + (\text{reactive volt-amperes})^2} \end{aligned}$$

When the current is conjugated, the reactive power is positive when the current lags the voltage and is negative when the current leads the voltage. These signs for the reactive power correspond to those obtained when the voltage is resolved into active and reactive components with respect to the current, as is customary when dealing with series circuits. When the voltage is conjugated, the reactive power is positive for leading current and

negative for lagging current. These signs are the same as are obtained when the current is resolved into active and reactive components with respect to voltage, as is customary when dealing with circuits with branches in parallel, such as ordinary power circuits.

Although power is not actually a vector, it is sometimes convenient in handling problems involving both active and reactive power to consider power as a vector in the sense given by equation (27). Equation (27) is what would be obtained by multiplying the complex expression for voltage referred to current as a reference axis by the magnitude of the current. When the voltage is conjugated, the expression for vector power is the same as would be obtained by multiplying the complex expression for the current referred to the voltage as a reference axis by the magnitude of the voltage.

Example of the Calculation of Power, Power Factor Etc.—Let an alternating voltage $\bar{E} = 100 + j50$ be impressed on a circuit whose constants are such that the current resulting is $\bar{I} = 20 - j30$. $\bar{E}$ and $\bar{I}$ are both root-mean-square or effective values.

$$E = \sqrt{(100)^2 + (50)^2} = 111.8 \text{ volts}$$

$$I = \sqrt{(20)^2 + (30)^2} = 36.06 \text{ amperes}$$

$$\text{Power} = P = (100) \times (20) + (50) \times (-30)$$
$$= 2000 - 1500 = 500 \text{ watts}$$

Let θ be the phase angle between the current and voltage.

$$\text{Power factor} = \cos\theta = \frac{P}{EI} = \frac{5000}{111.8 \times 36.06} = 0.1240$$

$$\theta = \cos^{-1}(0.1240) = 82.88 \text{ degrees}$$

$$\text{Reactive factor} = \sin\theta = 0.9923$$

$$\text{Reactive power} = EI \sin\theta$$
$$= 111.8 \times 36.06 \times 0.9923 = 3999 \text{ vars}$$

$$\text{Active component of current} = I\cos\theta$$
$$= 36.06 \times 0.1240 = 4.47 \text{ amperes}$$

$$\text{Reactive component of current} = 36.06 \times 0.9923 = 35.78 \text{ amperes}$$

The same results may be obtained by using the conjugate method.

$$\bar{E} = 100 + j50$$

$$\bar{I}_c \text{ (conjugate)} = 20 + j30$$

$$\text{Vector power } \bar{P} = P + jQ = \bar{E} \times \bar{I}_c$$

$$= (100 + j50)(20 + j30)$$

$$= 500 + j4000 \text{ vector watts}$$

$$\text{Active power} = 500 \text{ watts}$$

$$\text{Reactive power} = 4000 \text{ vars} \qquad \text{(lagging current)}$$

$$\text{Power factor} = \frac{P}{\sqrt{P^2 + Q^2}}$$

$$= \frac{500}{\sqrt{(500)^2 + (4000)^2}}$$

$$= 0.1240$$

$$\text{Reactive factor} = \frac{Q}{\sqrt{P^2 + Q^2}}$$

$$= \frac{4000}{\sqrt{(500)^2 + (4000)^2}}$$

$$= 0.9923$$

$$\text{Active component of current} = \frac{P}{V}$$

$$= \frac{500}{\sqrt{(100)^2 + (50)^2}} = 4.47 \text{ amperes}$$

$$\text{Reactive component of current} = \frac{Q}{V}$$

$$= \frac{4000}{\sqrt{(100)^2 + (50)^2}} = 35.78 \text{ amperes}$$

CHAPTER IV

NON-SINUSOIDAL WAVES

Wave Form of Alternators.—The wave form of commercial alternators is never exactly sinusoidal and under certain conditions it may differ therefrom considerably. The armature windings of multipolar alternators consist of as many identical elements per phase as there are pairs of poles. These elements are spaced 180 electrical degrees apart on the armature and are usually connected in series. Polyphase alternators have as many groups of windings as there are phases. These are displaced from one another by $\frac{360}{n}$ electrical degrees, where n is the number of phases except for two-phase. Since the groups of windings are always identical, the voltages induced in them are equal in magnitude but are displaced in time phase by $\frac{360}{n}$ degrees or by $\frac{1}{n}$ of the time of a complete cycle except for two-phase. The windings of a polyphase alternator are always interconnected in star or in mesh (see Chapter VIII) in order to diminish the number of terminals. In general, except in the case of single-phase and two-phase alternators, there are as many terminals as there are phases. Most commercial alternators are three-phase, although two-phase or four-phase alternators are occasionally used.

If the sides of the coils of an armature winding are 180 electrical degrees apart, and if one side of a coil occupies any given position with respect to a north pole, the other side of the coil occupies an exactly similar position with respect to the adjacent south pole. If the poles are similar so that the flux distribution produced by each is the same, the voltages induced in the two coil sides are equal in magnitude but opposite in direction. That is. if one acts from the front to the back of the coil, the other acts

from the back to the front. At every instant, the voltage generated in the coil is the difference between the voltages generated in its two sides.

The voltage generated in a coil side is equal to

$$e = \mathfrak{B}LsZ \times 10^{-8} \tag{1}$$

where L is the effective length of the coil side, *i.e.*, the length of the part cutting the flux, and Z is the number of inductors in each coil. An inductor is one of the two active sides of each turn of an armature coil. Each turn has two inductors. The inductors are the parts of an armature coil which cut flux. $\mathfrak{B}$ is the flux density in gausses at the inductor and s is the component of the velocity of the inductor perpendicular to the flux density. The velocity must be in centimeters per second.

The velocity s is fixed by the radius of the armature, the frequency and the number of poles. The only variable is the flux density $\mathfrak{B}$ over the pole face. This is determined by the total flux per pole and the shape of the pole face. Since the only variable is the flux density, the shape of the voltage wave must be the same as the shape of the curve of flux distribution in the air gap. If the pole faces are concentric with the armature surface, the flux distribution in the air gap is nearly constant over the pole faces, except for ripples produced by the armature slots, and drops rapidly to zero at points midway between poles. If the edges of the poles are chamfered so that the air gap at the edges of the pole is longer than at the middle, the flux density is no longer uniform over the pole face but is greatest at the center. By properly shaping the pole face, the flux density may be made to vary approximately sinusoidally in passing from a point midway between any pair of poles to a point midway between the next pair. If it could be made exactly sinusoidal, the voltage generated in each coil side would also be sinusoidal, since it is proportional to the flux density at each instant. If the pole faces are nearly concentric with the armature, the total flux for a given permissible maximum flux density is greater than when the pole faces are chamfered, but the voltage wave is flat. If the pole faces are chamfered too much, the wave form is peaked, *i.e.*, more peaked than a sinusoid. The presence of the armature

slots causes a periodic variation in the flux which produces high-frequency harmonics in the voltage. These may be much diminished by certain devices.

Even if the poles of an alternator could be shaped to give an exactly sinusoidal flux distribution at no load, the wave form of the alternator under load might differ considerably from a sinusoid, due to the effect of armature reaction in distorting the flux distribution. In general, except when the current is in quadrature with the induced voltage, armature reaction tends to crowd the flux toward one side of each pole and to give rise to a distorted voltage wave which may differ greatly from a sinusoid. Although the flux distribution may become badly distorted under load, it is possible to arrange an armature winding in such a way that the voltage between its terminals is still nearly sinusoidal.

Nearly all alternators have distributed armature windings. That is, the coils for any one phase are distributed among several pairs of slots per pair of poles instead of being placed in a single pair of slots per pair of poles. In this case the voltage induced in the coils is not in phase, but differs in phase by an angle equal to the angle in electrical degrees between the slots. Also, the coil pitch is often less than 180 electrical degrees, *i.e.*, the coil sides are less than 180 electrical degrees apart. A common pitch is $\frac{5}{6} \times 180 = 150$ electrical degrees.

One important effect of distributing a winding and also of shortening its pitch is to smooth out the wave form and make it more nearly sinusoidal by reducing or eliminating the components in the voltage which cause it to differ from a sinusoid. As will be explained later, any distorted voltage wave may be resolved into a series of sinusoidal components having frequencies which are 1, 2, 3, 4, 5 etc. times the fundamental frequency, *i.e.*, the frequency determined by the speed and the number of poles. The components having frequencies of 2, 4, 6 etc. times the fundamental frequency do not occur in the voltage waves of alternators, since these machines have symmetrical voltage waves. (See page 83.)

If the angle between the slots is α electrical degrees, the voltage generated in coils which are in adjacent slots are out of

phase by α degrees for the component of fundamental frequency, 3α degrees for the component of third frequency, 5α for the component of fifth frequency etc. The effect of this difference in phase is to reduce the distorting components in the voltage wave and thus to make the voltage more nearly sinusoidal. A similar effect is produced by shortening the pitch.

By the use of a suitably shortened pitch, any one component in the voltage wave may be completely eliminated in the coil voltage. For example, if the coil sides are $\frac{4}{5} \times 180 = 144$ electrical degrees apart, the components of fifth frequency in the two sides of any coil are $144 \times 5 = 720$ degrees out of phase. They are displaced two whole wave lengths and are therefore in phase. Since the coil voltage is the vector difference between the voltages generated in the two coil sides, a $\frac{4}{5}$ pitch completely eliminates the voltage component of fifth frequency. Any component may also be eliminated by distributing the winding among a suitable number of properly spaced slots.

The triple-frequency component in the voltage is eliminated in a three-phase alternator by the interconnection of the phases.

Instead of trying to eliminate certain of the distorting components completely by the use of a suitable coil pitch coupled with the distribution of the winding, a pitch and a distribution are usually chosen which considerably reduce a number of the distorting components without eliminating any one of them completely. Better average wave form may be obtained in this way than when certain components are completely eliminated.

Although it is possible to design an alternator, by the use of shortened pitch, a distributed winding and certain other devices, to give practically a sinusoidal wave of voltage under load as well as at no load, alternators are not commercially so designed except when they are to be used for special work, such as cable testing, which requires sinusoidal voltage. Under ordinary conditions, the wave form of commercial alternators does not differ greatly from a sinusoid. Even if the voltage wave of an alternator were exactly sinusoidal under all conditions, the current wave would not necessarily be sinusoidal. This may be caused by a periodic variation in the constants of the load, or

by the wave shape of the electromotive force of the load, in case it is a motor. In general, inductance in the load smooths out the current obtained from an alternator with a distorted wave. Capacitance in the load has the opposite effect, *i.e.*, it accentuates the distortion in the current wave. It is, therefore, necessary to consider the conditions existing when the voltage and current waves are not simple harmonic functions of the time.

Representation of a Non-sinusoidal Current or Voltage Wave by a Fourier Series.—Any single-valued periodic function may be resolved into a Fourier series[1] consisting of sine and cosine terms of different relative magnitudes and with frequencies which are in the ratios of 1, 2, 3, 4, 5 etc. to the frequency of the fundamental period of the function. For example, any single-valued periodic quantity, like the voltage or current of an alternator, may be represented by the following series. For the alternator, A_0 is zero.

$$e = A_0 + A_1 \sin \omega t + B_1 \cos \omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + A_3 \sin 3\omega t + B_3 \cos 3\omega t + \text{etc.} \quad (2)$$

The sine and cosine terms are called *harmonics*. The first harmonics are called *fundamentals*. Theoretically, an infinite number of terms is required to represent most non-sinusoidal voltages or currents, but for most practical purposes the first few terms are sufficient. It is seldom necessary to go beyond the term of eleventh frequency. For most wave forms the series converges rapidly.

Under ordinary conditions, the first harmonic or fundamental (this includes both the sine and cosine terms of fundamental frequency) is the most important term in the series and it has by far the largest amplitude. Any harmonic may be absent from a voltage or current wave. On the other hand, certain conditions may very much exaggerate or diminish a harmonic in a current wave. Also, certain harmonics may be present in a current wave which are not present in the voltage causing that wave. This always occurs when a voltage is impressed on a circuit whose resistance, inductance or capacitance is a function of the current.

[1] See Fourier's Series and Spherical Harmonics, Byerly.

The sine and cosine terms of the Fourier series may be combined to give a series involving either sine or cosine terms alone. For example, equation (2) may be written

$$e = A_0 + C_1 \sin(\omega t + \theta_1) + C_2 \sin(2\omega t + \theta_2) + C_3 \sin(3\omega t + \theta_3) + \text{etc.} \quad (3)$$

or

$$e = A_0 + C_1 \cos(\omega t - \theta_1') + C_2 \cos(2\omega t - \theta_2') + C_3 \cos(3\omega t - \theta_3') + \text{etc.} \quad (4)$$

The angles θ in equation (3) are the angles of lead between the harmonics of the sine series and the corresponding sine components in equation (2). The angles θ' in equation (4) are the angles of lag between the harmonics of the cosine series and the corresponding cosine components in equation (2). All the angles θ and θ' in equations (3) and (4) are measured on the scales of angles for the harmonics. A phase displacement of θ degrees for the nth harmonic corresponds to a displacement of $\frac{\theta}{n}$ degrees for the fundamental. (See Fig. 22.) If the phase angles for the harmonics are measured on the scale of angles for the fundamental, equations (3) and (4) become

$$e = A_0 + C_1 \sin(\omega t + \alpha_1) + C_2 \sin 2(\omega t + \alpha_2) + C_3 \sin 3(\omega t + \alpha_3) + \text{etc.} \quad (5)$$

$$e = A_0 + C_1 \cos(\omega t - \alpha_1') + C_2 \cos 2(\omega t - \alpha_2') + C_3 \cos 3(\omega t - \alpha_3') + \text{etc.} \quad (6)$$

where $\alpha_1 = \theta_1$, $\alpha_1' = \theta_1'$, $\alpha_2 = \frac{1}{2}\theta_2$, $\alpha_2' = \frac{1}{2}\theta_2'$, $\alpha_3 = \frac{1}{3}\theta_3$ and $\alpha_3' = \frac{1}{3}\theta_3'$. In general, $\alpha_n = \frac{1}{n}\theta_n$ and $\alpha_n' = \frac{1}{n}\theta_n'$, where n is the order of the harmonic considered.

Since sine and cosine functions of time differ in phase by 90 degrees, the cosine function leading, the A and B terms in equation (2) are in quadrature. Therefore,

$$C_1 = \sqrt{A_1^2 + B_1^2} \qquad \tan\theta_1 = \frac{B_1}{A_1} \qquad \tan\theta_1' = \frac{A_1}{B_1}$$

$$C_2 = \sqrt{A_2^2 + B_2^2} \qquad \tan\theta_2 = \frac{B_2}{A_2} \qquad \tan\theta_2' = \frac{A_2}{B_2}$$

In general,

$$C_n = \sqrt{A_n^2 + B_n^2} \qquad \tan \theta_n = \frac{B_n}{A_n} \qquad \tan \theta_n' = \frac{A_n}{B_n}$$

Since, ordinarily, the current and voltage waves of alternators operating under steady conditions are symmetrical with respect to the axis of time, the constant A_0 in the Fourier series for a current or a voltage may usually be omitted.

A wave containing a fundamental and a third and a seventh harmonic is shown in Fig. 22. The fundamental and the harmonics are shown by full lines. The resultant wave is shown dotted.

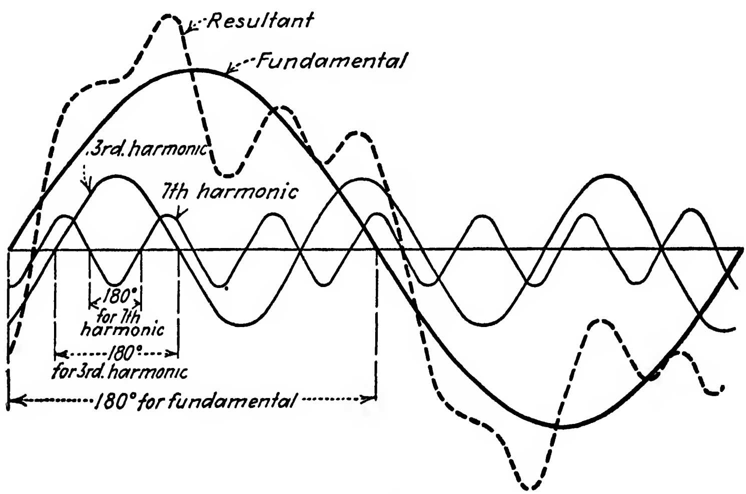


FIG. 22.

Effect of Even Harmonics on Wave Form.—Symmetrical waves, *i.e.*, waves which have exactly similar positive and negative loops, cannot contain even harmonics, since the phase of any even harmonic with respect to the fundamental is opposite in the two halves of the wave. Let the following Fourier series represent a voltage which contains both odd and even harmonics:

$$e = E_1 \sin (\omega t + \theta_1) + E_2 \sin (2\omega t + \theta_2) + E_3 \sin (3\omega t + \theta_3) + E_4 \sin (4\omega t + \theta_4) + E_5 \sin (5\omega t + \theta_5) + \text{etc.} \qquad (7)$$

If this wave is symmetrical with respect to its positive and negative loops, its instantaneous values at two instants of time, such as t and $\left(t + \frac{T}{2}\right) = \left(t + \frac{1}{2f}\right)$, which are separated by one-half period, must be equal in magnitude but opposite in sign. Equation (7) gives the instantaneous value of e for a time $t = t$.

For $t = \left(t + \frac{1}{2f}\right)$, the instantaneous value of e is

$$\begin{aligned} e' = E_1 \sin\left(\omega t + \frac{\omega}{2f} + \theta_1\right) &+ E_2 \sin\left(2\omega t + \frac{\omega}{f} + \theta_2\right) \\ &+ E_3 \sin\left(3\omega t + \frac{3\omega}{2f} + \theta_3\right) + E_4 \sin\left(4\omega t + \frac{2\omega}{f} + \theta_4\right) \\ &+ E_5 \sin\left(5\omega t + \frac{5\omega}{2f} + \theta_5\right) + \text{etc.} \end{aligned} \tag{8}$$

Remembering that $\omega = 2\pi f$ and also that a phase displacement of any whole number of wave lengths, *i.e.*, any whole number of 2π radians, is equivalent to zero displacement so far as phase relations are concerned, equation (8) may be reduced to the following form:

$$\begin{aligned} e' &= E_1 \sin(\omega t + \pi + \theta_1) + E_2 \sin(2\omega t + \theta_2) \\ &\quad + E_3 \sin(3\omega t + \pi + \theta_3) + E_4 \sin(4\omega t + \theta_4) \\ &\quad + E_5 \sin(5\omega t + \pi + \theta_5) \\ &= -E_1 \sin(\omega t + \theta_1) + E_2 \sin(2\omega t + \theta_2) \\ &\quad - E_3 \sin(3\omega t + \theta_3) + E_4 \sin(4\omega t + \theta_4) \\ &\quad - E_5 \sin(5\omega t + \theta_5) + \text{etc.} \end{aligned} \tag{9}$$

It will be seen, by comparing equations (7) and (9), that, while the fundamentals and the corresponding odd harmonics for points one-half a period apart are opposite in phase, the even harmonics are in phase. Therefore, the two halves of a wave containing even harmonics cannot be alike in shape.

That the two halves of a wave which contains even harmonics are not alike, *i.e.*, that a wave containing even harmonics is not symmetrical, is also shown by Fig. 23, which shows a wave containing a fundamental and a second harmonic.

Since the voltage waves of commercial alternators are symmetrical, they cannot contain even harmonics. Even harmonics

may, however, occur in a current wave, although they are not present in the voltage producing it. Even harmonics occur in the current wave when an alternating voltage is impressed on the winding of an inductive circuit having an iron core which is magnetized in a fixed direction by any means, such as a constant direct current in a second winding surrounding the core. In this case, the self-induction of the winding carrying the alternating current is a function of the current and is different for positive and negative values of the alternating current. This should be better understood after studying Chapter V. In general, even harmonics occur in rectifier and vacuum-tube circuits.

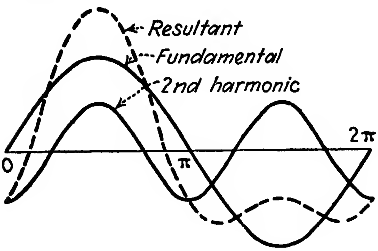


FIG. 23.

Waves Which Have the Halves of the Positive and Negative Loops Symmetrical.—Waves which are symmetrical with respect to their positive and negative loops cannot contain even harmonics. If the halves of the positive and of the negative loops are also to be symmetrical, the fundamental and all harmonics must pass through zero at the same instant. For this condition to be fulfilled, either the sine or the cosine terms in the Fourier series given in equation (2) must be zero. All even harmonics and the constant A_0 must also be zero if the positive and negative loops are to be symmetrical.

If the equation for the wave is written in sine or cosine terms only [see equations (3) and (4), page 81], the phase angles of all harmonics must be either zero or 180 degrees when the phase angle for the fundamental is made zero by considering time t zero when the fundamental is zero.

Changing the Reference Point from Which Angles and Time Are Measured in a Complex Wave.—It is frequently convenient and often necessary, when considering complex waves, to change the position point from which angles and time are reckoned. One case in which this is necessary is when the wave forms of two waves, whose Fourier equations are known, are to be compared.

For two waves to be similar, they must not only contain like harmonics, but the relative magnitudes of the harmonics and fundamental and their phase relations must be alike in the two waves.

A glance at the Fourier equations of any two waves, which are to be compared for wave form, tells whether they contain the same harmonics. The relative magnitudes of the harmonics and the fundamental may easily be determined from the coefficients of the Fourier series, but, unless the points from which angles are measured occupy the same positions with respect to the fundamentals, the relative phase of the harmonics in the waves is not obvious.

If the points from which angles are reckoned do not occupy the same relative positions with respect to the fundamentals, it is necessary to shift one of them until they do.

Consider the following two waves:

$$e_1 = 100 \sin (\omega t + 30°) + 50 \sin (2\omega t + 45°) + 40 \sin (3\omega t + 75°) \quad (10)$$

$$e_2 = 150 \sin (\omega t - 10°) + 75 \sin (2\omega t - 35°) + 60 \sin (3\omega t - 45°) \quad (11)$$

The waves contain like harmonics. The magnitudes of the fundamental and harmonics in the second wave are fifty per cent greater than in the first wave. The relative magnitudes of the harmonics and fundamental are the same in both waves.

To compare the phase relations of the harmonics and fundamentals, one wave must be shifted in phase by an amount equal to the difference between the fundamental phase angles of the two waves. This can be accomplished either by shifting the first wave minus 40 degrees or the second plus 40 degrees. Shift the second wave. Since any phase displacement of α degrees for the fundamental corresponds to a phase displacement of $n\alpha$ degrees for the nth harmonic, it follows that, if 40 degrees is added to the fundamental phase angle, $2 \times 40 = 80$ degrees must be added to the phase angle of the second harmonic and $3 \times 40 = 120$ degrees must be added to the phase angle of the third harmonic. Shifting the fundamental of the second wave plus 40 degrees in phase gives

$$e_2' = 150 \sin (\omega t + 30°) + 75 \sin (2\omega t + 45°) + 60 \sin (3\omega t + 75°) \qquad (12)$$

From equations (10) and (12) it is obvious that the differences in phase between the harmonics and fundamental are alike in the two waves.

Fourier Series for Rectangular and for Triangular Waves.—The Fourier equations for a rectangular and for a triangular wave are interesting since such waves represent extreme cases of a flattened and a peaked sine wave. Neither of these wave forms could be

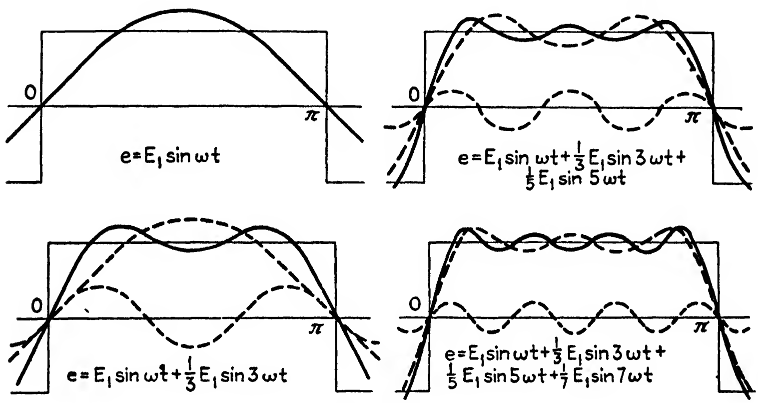


Fig. 24.

exactly attained in practice nor would either be desirable even if it could be secured.

The Fourier series for rectangular and for triangular voltage waves are given by equations (13) and (14), respectively.

$$e = E_1 \sin \omega t + \frac{1}{3} E_1 \sin 3\omega t + \frac{1}{5} E_1 \sin 5\omega t + \frac{1}{7} E_1 \sin 7\omega t + \text{etc.} \qquad (13)$$

$$e = E_1' \sin \omega t - \frac{1}{3^2} E_1' \sin 3\omega t + \frac{1}{5^2} E_1' \sin 5\omega t - \frac{1}{7^2} E_1' \sin 7\omega t + \text{etc.} \qquad (14)$$

Both the rectangular and the triangular waves have similar positive and negative loops and, therefore, contain only odd

harmonics. Also, since the halves of each loop are similar, all the phase angles are zero. The height of the rectangular wave is $\frac{\pi}{4}E_1$. The peak of the triangular wave is $2E_1$.

The manner in which the wave form given by equation (13) approaches that of a rectangular wave, as successive terms are added, is illustrated in Fig. 24.

Although an infinite number of terms would be required to represent a rectangular wave exactly, a fairly good approximation to such a wave is obtained with comparatively few terms. Even with the four terms plotted in Fig. 24, the resultant is rapidly flattening out and approaching the rectangular form. Ordinary waves met in practice can usually be represented with sufficient accuracy for most work by comparatively few terms of their Fourier equations.

Measurement of Current, Voltage and Power When the Wave Form Is Not Sinusoidal.—Since the average torque producing the deflection in the electrodynamometer and iron-vane types of ammeter is proportional to the average square of the current in their coils, such instruments may be used to measure current when the wave form is not sinusoidal. This statement is not strictly correct for the iron-vane type of instrument, since the eddy currents and hysteresis in the iron vane may seriously affect the readings when pronounced high-frequency harmonics are present. However, when used on circuits of commercial frequencies and wave forms, the readings are approximately correct. Voltmeters are usually of the electrodynamometer type. Iron vanes are seldom used, except in the cheaper instruments. For an electrodynamometer type of voltmeter to indicate correctly, the current through it at each instant must be strictly proportional to the instantaneous voltage across its terminals.

The general effect of inductance in a circuit is to damp out harmonics in the current caused by those in the voltage to a greater and greater degree as their frequency increases. Therefore, since the coils of an instrument cannot be made without inductance, the current in a voltmeter cannot be strictly proportional, at each instant, to the voltage across its terminals, except when the voltage is sinusoidal. If the non-inductive resistance

in series with a voltmeter is large compared with the inductance of its coils, the current through it is substantially proportional, at each instant, to the voltage across its terminals, provided the voltage does not contain very pronounced harmonics of high frequency. Both the electrodynamometer and iron-vane types of voltmeter, when used on circuits of commercial frequencies and wave forms, indicate root-mean-square or effective voltage to a high degree of precision.

For high frequencies, the thermocouple type of instrument is used. Such an instrument consists of a sensitive direct-current milliammeter, a thermocouple and a heater which is joined to the thermocouple by welding and carries the current to be measured. The milliammeter measures the current caused by the thermocouple. Since the heating of the heating element is proportional to the average square of the current, such an instrument indicates effective current independently of its frequency or wave form. The scale of the instrument is not uniform. The hot-wire type of instruments has been used for measuring high-frequency currents.' Such instruments do not hold their calibration well and are not satisfactory. The hot-wire instrument depends on the elongation of a wire which is heated by the current to be measured.

The electrodynamometer type of wattmeter indicates true average power, provided the current in its current coil is equal or proportional, at each instant, to the current in the circuit, and provided the current in its potential circuit is proportional, at each instant, to the voltage across its terminals. The instrument is subject to the same limitations as the voltmeter, but by proper design it may be made to indicate true average power when used in circuits of ordinary commercial frequency and wave form.

Determination of Wave Form.—The usual way of determining the wave form of a current or voltage is by means of an oscillograph. The oscillograph, in its simplest form, consists of a vibrating element made of a single loop of very fine wire which is stretched over two bridges. The straight, parallel sides of the loop between the bridges are from 0.2 to 0.25 millimeter apart, and lie in a strong, uniform field between the poles of an electromagnet which is excited with direct current. The plane of the

loop is parallel to the axis of the magnetic field. A very small, light mirror is attached to the parallel sides of the loop midway between the bridges.

When a current is passed through the loop, it flows down one side and up the other, causing the two sides of the loop to deflect in opposite directions. The mirror tilts through an angle which is proportional to the strength of the current. If a spot of light from a source which approximates a point is reflected on a screen by the mirror, the spot moves, when the mirror is deflected, through a distance which is proportional to angular displacement of the mirror. This assumes that the displacement of the mirror is small. The loop with its mirror is immersed in oil to damp its vibrations and make it dead beat. When the wave form is to be photographed, the spot of light is reflected on a revolving drum which carries the film. The rotation of this drum gives the time element, *i.e.*, the abscissa of the wave, and the displacement of the spot, which must be parallel to the axis of the drum, gives the other element, *i.e.*, the ordinate of the wave. To prevent overlapping of the waves on the drum, a shutter is interposed between the light source and the drum and remains open during only one revolution of the drum. When the wave is to be observed or traced on a screen, the time element of the wave is obtained by placing a revolving mirror between the source of light and the screen. To keep the image of the wave fixed on the screen, this mirror must revolve synchronously with the frequency of the circuit.

The dimensions of the vibrating system must be such that its natural frequency of vibration is at least fifty times as great as the frequency of the highest harmonic to be detected. The minimum free period which it is practical to obtain is about 0.0001 second, but for most commercial work a period of 0.0002 second is sufficiently low.

For current measurements, the vibrating element is shunted across the terminals of a non-inductive shunt which is placed in series with the circuit carrying the current whose wave form is to be determined. Since the vibrating element is sensibly non-inductive, the current it carries at each instant is directly proportional to the current in the circuit. For voltage measurements, the vibrating element is connected in series with a suitable

non-inductive resistance and is then shunted across the circuit whose voltage wave form is to be determined.

On account of the inertia of the type of oscillograph just described, harmonics of high frequency cannot be recorded accurately. For this reason, its use is limited to power circuits of ordinary frequencies. It also requires a relatively large current, about 0.1 ampere, for its satisfactory operation. For very high frequencies as well as for small currents, the cathode-ray oscillograph is used. In this instrument, a stream of electrons is made to serve for the vibrating element. This stream is made to impinge on either a fluorescent screen for viewing the image or on a photographic plate if a permanent record is desired.

Effective Value of a Non-sinusoidal Electromotive Force or Current.—Let

$$e = E_{m1} \sin (\omega t + \theta_1) + E_{m2} \sin (2\omega t + \theta_2) + E_{m3} \sin (3\omega t + \theta_3) + \text{etc.} \tag{15}$$

be an alternating electromotive force of periodic time $T = \frac{2\pi}{\omega}$, containing a fundamental, $E_{m1} \sin (\omega t + \theta_1)$, and both even and odd harmonics, $E_{m2} \sin (2\omega t + \theta_2)$, $E_{m3} \sin (3\omega t + \theta_3)$ etc., whose periods are multiples of the fundamental period $\frac{2\pi}{\omega}$. The effective or root-mean-square value of the electromotive force is given by

$$E = \sqrt{\frac{1}{T}\int_0^T e^2 dt}$$

$$= \sqrt{\frac{1}{T}\int_0^T \{E_{m1} \sin (\omega t + \theta_1) + E_{m2} \sin (2\omega t + \theta_2) + E_{m3} \sin (3\omega t + \theta_3) + \text{etc.}\}^2 dt} \tag{16}$$

This involves squared terms of the general form

$$\frac{1}{T}\int_0^T E_{mk}^2 \sin^2 (k\omega t + \theta_k) dt$$

and product terms of the general form

$$\frac{1}{T}\int_0^T \{E_{mk} \sin (k\omega t + \theta_k) \times E_{mq} \sin (q\omega t + \theta_q)\} dt$$

Each of the squared terms, on integration over a complete cycle, becomes equal to one-half the square of its maximum value. Each of the product terms of unlike frequency becomes zero on integration over a complete cycle, since the average value of the product of two sine terms of unlike frequency is zero.

Consider the product of two sine terms of like frequency. Let $\omega t = \alpha$.

$$\begin{aligned}\mathrm{f}(\alpha) &= A \sin(\alpha + \theta) \times B \sin(\alpha + \theta') \\ &= AB(\sin\alpha\cos\theta + \cos\alpha\sin\theta)(\sin\alpha\cos\theta' + \cos\alpha\sin\theta') \\ &= AB(\sin^2\alpha\cos\theta\cos\theta' + \sin\alpha\cos\alpha\cos\theta\sin\theta' \\ &\qquad + \cos\alpha\sin\alpha\sin\theta\cos\theta' + \cos^2\alpha\sin\theta\sin\theta')\end{aligned}$$

But

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}, \quad \cos^2\alpha = \frac{1 + \cos 2\alpha}{2}, \quad \sin\alpha\cos\alpha = \frac{\sin 2\alpha}{2}$$

Therefore,

$$\begin{aligned}\mathrm{f}(\alpha) = AB\biggl(&\frac{1 - \cos 2\alpha}{2}\cos\theta\cos\theta' + \frac{\sin 2\alpha}{2}\cos\theta\sin\theta' \\ &+ \frac{\sin 2\alpha}{2}\sin\theta\cos\theta' + \frac{1 + \cos 2\alpha}{2}\sin\theta\sin\theta'\biggr)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{f}(\alpha)d\alpha &= AB\left(\frac{1}{2}\cos\theta\cos\theta' + 0 + 0 + \frac{1}{2}\sin\theta\sin\theta'\right) \\ &= \frac{AB}{2}\cos(\theta - \theta') \qquad (17)\end{aligned}$$

If the two terms are identical,

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{f}(\alpha) = \frac{A^2}{2} = \frac{B^2}{2}$$

Now consider the product of two sine terms of unlike frequency.

$$\begin{aligned}\mathrm{f}(\alpha) &= A \sin(\alpha + \theta) \times B \sin(n\alpha + \theta_n) \\ &= AB\{(\sin\alpha\cos\theta + \cos\alpha\sin\theta)(\sin n\alpha\cos\theta_n \\ &\qquad\qquad + \cos n\alpha\sin\theta_n)\} \\ &= AB(\sin\alpha\sin n\alpha\cos\theta\cos\theta_n + \sin\alpha\cos n\alpha\cos\theta\sin\theta_n \\ &\qquad + \cos\alpha\sin n\alpha\sin\theta\cos\theta_n + \cos\alpha\cos n\alpha\sin\theta\sin\theta_n)\end{aligned}$$

where n is any integer which is greater than unity.

Since θ and θ_n are constants,

$$\mathrm{f}(\alpha) = AB(K_1 \sin \alpha \sin n\alpha + K_2 \sin \alpha \cos n\alpha + K_3 \cos \alpha \sin n\alpha + K_4 \cos \alpha \cos n\alpha)$$

But

$$\sin x \sin y = \frac{1}{2}\{\cos (x - y) - \cos (x + y)\}$$

$$\sin x \cos y = \frac{1}{2}\{\sin (x + y) + \sin (x - y)\}$$

$$\cos x \sin y = \frac{1}{2}\{\sin (x + y) - \sin (x - y)\}$$

$$\cos x \cos y = \frac{1}{2}\{\cos (x + y) + \cos (x - y)\}$$

Therefore,

$$\begin{aligned}\mathrm{f}(\alpha) = AB\Big\{&\frac{K_1}{2}[\cos (1 - n)\alpha - \cos (1 + n)\alpha] \\ &+ \frac{K_2}{2}[\sin (1 + n)\alpha + \sin (1 - n)\alpha] \\ &+ \frac{K_3}{2}[\sin (1 + n)\alpha - \sin (1 - n)\alpha] \\ &+ \frac{K_4}{2}[\cos (1 + n)\alpha + \cos (1 - n)\alpha]\Big\}\end{aligned}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{f}(\alpha)d\alpha = 0 \qquad (18)$$

The root-mean-square or effective value of a non-sinusoidal voltage is, therefore,

$$E = \sqrt{\frac{E_{m1}^2 + E_{m2}^2 + E_{m3}^2 + \text{etc.}}{2}} \qquad (19)$$

$$= \sqrt{E_1^2 + E_2^2 + E_3^2 + \text{etc.}} \qquad (20)$$

where the E's with the subscript m are maximum values of the components of the voltage. Without the subscript m they are the effective values. Similarly, if

$$i = I_{m1} \sin (\omega t + \theta_1 + \alpha_1) + I_{m2} \sin (2\omega t + \theta_2 + \alpha_2) + I_{m3} \sin (3\omega t + \theta_3 + \alpha_3) + \text{etc.}$$

is an alternating current of periodic time $T = \frac{2\pi}{\omega}$, its root-mean-square or effective value is given by

$$I = \sqrt{\frac{I_{m1}^2 + I_{m2}^2 + I_{m3}^2 + \text{etc.}}{2}}$$
$$= \sqrt{I_1^2 + I_2^2 + I_3^2 + \text{etc.}} \qquad (21)$$

where the I's with the subscript m are maximum values of the components of the current. Without the subscript m they are the effective values.

Power when the Electromotive Force and Current Are Non-sinusoidal Waves.—Let

$$e = E_{m1} \sin(\omega t + \theta_1) + E_{m2} \sin(2\omega t + \theta_2) + E_{m3} \sin(3\omega t + \theta_3) + \text{etc.}$$

and

$$i = I_{m1} \sin(\omega t + \theta_1') + I_{m2} \sin(2\omega t + \theta_2') + I_{m3} \sin(3\omega t + \theta_3') + \text{etc.}$$

represent an electromotive force and current, respectively, whose periodic time is $T = \frac{2\pi}{\omega}$. The average power is

$$P = \frac{1}{T}\int_0^T eidt$$
$$= \frac{1}{T}\int_0^T \{E_{m1} \sin(\omega t + \theta_1) + E_{m2} \sin(2\omega t + \theta_2) + E_{m3} \sin(3\omega t + \theta_3) + \text{etc.}\} \times \{I_{m1} \sin(\omega t + \theta_1') + I_{m2} \sin(2\omega t + \theta_2') + I_{m3} \sin(3\omega t + \theta_3') + \text{etc.}\}dt$$

This involves product terms of like and unlike frequency of the forms

$$\frac{1}{T}\int_0^T \{E_{mk} \sin(k\omega t + \theta_k) \times I_{mk} \sin(k\omega t + \theta_k')\}dt$$

and

$$\frac{1}{T}\int_0^T \{E_{mk} \sin(k\omega t + \theta_k) \times I_{mq} \sin(q\omega t + \theta_q')\}dt$$

On integration, the product terms of like frequency become [see equation (17), page 91]

$$\frac{E_{mk}I_{mk}}{2} \cos (\theta_k - \theta_k')$$

while the product terms of unlike frequency become zero on integration. [See equation (18), page 92.]

The average power in a circuit having harmonics in both current and voltage is, therefore,

$$P = \frac{E_{m1}I_{m1}}{2} \cos (\theta_1 - \theta_1') + \frac{E_{m2}I_{m2}}{2} \cos (\theta_2 - \theta_2') + \frac{E_{m3}I_{m3}}{2} \cos (\theta_3 - \theta_3') + \text{etc.} \quad (22)$$

$$P = E_1I_1 \cos (\theta_1 - \theta_1') + E_2I_2 \cos (\theta_2 - \theta_2') + E_3I_3 \cos (\theta_3 - \theta_3') + \text{etc.} \quad (23)$$

The letters E and I in equations (22) and (23) with the subscript m represent maximum values. Without the subscript m they represent root-mean-square values.

If a harmonic occurs in the current and is not present in the voltage, or, *vice versa,* if a harmonic occurs in the voltage and is not present in the current, it contributes nothing to the average power developed.

Power Factor when the Current and Voltage Are not Sinusoidal.—The power factor of a circuit is defined as the ratio of the true average power to the volt-amperes. This definition is independent of wave form.

$$\text{Power factor} = \frac{\text{true power}}{\text{volt-amperes}} = \frac{P}{EI} \quad (24)$$

$$= \frac{E_1I_1 \cos (\theta_1 - \theta_1') + E_2I_2 \cos (\theta_2 - \theta_2') + \text{etc.}}{\sqrt{E_1^2 + E_2^2 + \text{etc.}} \times \sqrt{I_1^2 + I_2^2 + \text{etc.}}} \quad (25)$$

Although a harmonic which occurs in the current of a circuit and does not occur in the voltage does not contribute to the average power, it does increase the root-mean-square or effective value of the current required to produce the power. Therefore, the power factor of a circuit containing a harmonic in its current which is not present in its voltage cannot be unity. Similarly, the power factor of a circuit containing a harmonic in its voltage which is not present in its current cannot be unity.

The only way the power factor can be unity is for

$$\cos(\theta_1 - \theta_1') = \cos(\theta_2 - \theta_2') = \text{etc.} = 1$$

and

$$\frac{E_1}{I_1} = \frac{E_2}{I_2} = \text{etc.}$$

When the current and voltage waves are of different form, the maximum power factor, for fixed effective value of current and voltage, obviously occurs when the phase displacement between the current and voltage is that which makes the power a maximum.

From equation (25) it is obvious that unity power factor can occur only when the current and voltage waves are exactly similar in form and have no phase displacement with respect to each other. In other words, both waves must contain like harmonics and these harmonics must have the same relative magnitudes and the same relative phase relations. If there is a harmonic in the current which is not present in the voltage, the power factor cannot be unity, since this harmonic contributes to the root-mean-square value of the current without adding to the power developed. A similar statement is true regarding a harmonic in the voltage.

There are many cases where harmonics are present in the current and are not present in the voltage causing it. Such a condition always occurs when a voltage is impressed on a circuit whose inductance is a function of the current. Since the inductance of a circuit is defined as the flux linkages per unit current, *i.e.*, as $L = N\frac{d\varphi}{di}$, the inductance of all circuits containing iron must be a function of the current.

Since the power factor of a circuit is equal to the ratio of true power to volt-amperes, power factor might be defined as the ratio of the actual power to the maximum power that could be obtained with the given current and voltage.

Example of the Calculation of Effective Values of Current and Voltage, Average Power and Power Factor for a Circuit when the Fourier Equations of Its Current and Voltage Are Known.— When the voltage

$$v = 200 \sin (377t + 10°) + 75 \sin (1131t + 30°) + 50 \sin (1885t + 50°)$$

is impressed on a certain circuit, the current is

$$i = 8.51 \sin (377t + 11°.18) + 8.04 \sin (1131t + 74°.33) + 7.50 \sin (1885t + 84°.32)$$

$$V = \sqrt{\frac{(200)^2 + (75)^2 + (50)^2}{2}}$$
$$= 155.1 \text{ volts}$$
$$I = \sqrt{\frac{(8.51)^2 + (8.04)^2 + (7.50)^2}{2}}$$
$$= 9.84 \text{ amperes}$$
$$P = \frac{200 \times 8.51}{2} \cos (10° - 11°.18) + \frac{75 \times 8.04}{2} \cos (30° - 74°33) + \frac{50 \times 7.50}{2} \cos (50° - 84°.32)$$
$$= 851 + 216 + 155 = 1222 \text{ watts}$$
$$\text{Power factor} = \frac{1222}{155.1 \times 9.84} = 0.801$$

Analysis of a Non-sinusoidal Wave and Determination of Its Fundamental and Harmonics.—It is often necessary to analyze a voltage or current wave, obtained by an oscillograph or other means, into the components of its Fourier series in order to determine the harmonics present and their magnitudes. The Fourier series may be written

$$f(\alpha) = A + A_1 \sin \alpha + B_1 \cos \alpha + A_2 \sin 2\alpha + B_2 \cos 2\alpha + \cdots + A_n \sin n\alpha + B_n \cos n\alpha$$

The expression for the constant A may be found by multiplying the equation by $d\alpha$ and then integrating it between the limits 0 and 2π. For example:

$$\int_0^{2\pi} f(\alpha)\, dx = A \int_0^{2\pi} d\alpha + A_1 \int_0^{2\pi} \sin \alpha d\alpha + B_1 \int_0^{2\pi} \cos \alpha d\alpha + \cdots + A_n \int_0^{2\pi} \sin n\alpha d\alpha + B_n \int_0^{2\pi} \cos n\alpha d\alpha$$

All terms on the right-hand side of the equation, except the first, reduce to zero leaving

$$\int_0^{2\pi} f(\alpha)d\alpha = 2\pi A$$

and

$$A = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)d\alpha$$

The integral $\int_0^{2\pi} f(\alpha)d\alpha$ is the net area enclosed by the curve represented by $f(\alpha)$ for a complete period or cycle. Since the positive and negative loops of current and voltage waves of alternators under steady conditions are symmetrical with respect to the axis of time, the net area represented by the integral and hence the constant A are zero.

To obtain the second constant, A_1, multiply both sides of the equation of the wave by sin $\alpha d\alpha$ and integrate the resulting equation between the limits 0 and 2π.

$$\begin{aligned}\int_0^{2\pi} f(\alpha) \sin \alpha d\alpha &= A\int_0^{2\pi} \sin \alpha d\alpha \\ &+ A_1\int_0^{2\pi} \sin^2 \alpha d\alpha + B_1\int_0^{2\pi} \cos \alpha \sin \alpha d\alpha \\ &+ A_2\int_0^{2\pi} \sin 2\alpha \sin \alpha d\alpha + B_2\int_0^{2\pi} \cos 2\alpha \sin \alpha d\alpha + \cdots \\ &+ A_n\int_0^{2\pi} \sin n\alpha \sin \alpha d\alpha + B_n\int_0^{2\pi} \cos n\alpha \sin \alpha d\alpha\end{aligned}$$

All the terms on the right-hand side of the equation, except the second, reduce to zero leaving

$$\int_0^{2\pi} f(\alpha) \sin \alpha d\alpha = \pi A_1$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \sin \alpha d\alpha$$

The constant B_1 may be found in a similar manner by multiplying the equation of the wave by cos $\alpha d\alpha$ and integrating between the limits 0 and 2π. In general, any constant may

be found by multiplying both sides of the equation by $d\alpha$ and the trigonometrical function which appears in the term containing the desired constant, and then integrating the resulting equation between the limits 0 and 2π. For example, any constant such as A_n may be found by multiplying both sides of the equation of the wave form by $\sin n\alpha d\alpha$ and then integrating the resulting

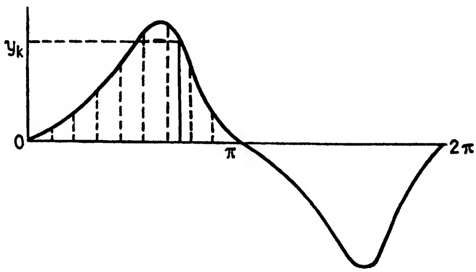


FIG. 25.

equation between the limits 0 and 2π. Doing this for the different terms gives

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \sin \alpha d\alpha$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \cos \alpha d\alpha$$

$$A_2 = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \sin 2\alpha d\alpha$$

$$B_2 = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \cos 2\alpha d\alpha$$

. . . .

. . . .

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \sin n\alpha d\alpha$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \cos n\alpha d\alpha$$

The values of the constants may be found by evaluating the integrals by means of a graphical method. Let the wave represented by $f(\alpha) = y$ be plotted as shown in Fig. 25. The base of this figure represents a complete cycle and is equal to 2π radians.

Divide the figure into m equal parts by equally spaced ordinates. The distance between any two adjacent ordinates is equal to $\frac{2\pi}{m}$ radians. The integral

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} f(\alpha) \sin \alpha d\alpha$$

may then be written as a summation.

$$A_1 = \frac{2}{m} \Sigma_1^m y_k \sin \left(k - \frac{1}{2}\right) \frac{2\pi}{m}$$

where y_k is the ordinate of the curve at the middle of the kth space.

$$A_1 = \frac{2}{m}\left\{y_1 \sin \left(\frac{1}{2}\frac{2\pi}{m} + y_2 \sin \frac{3}{2}\frac{2\pi}{m} + y_3 \sin \frac{5}{2}\frac{2\pi}{m} + \cdots \right.\right.$$
$$\left.+ y_m \sin \left(\frac{2m-1}{2}\right)\frac{2\pi}{m}\right\}$$

In general,

$$A_n = \frac{2}{m}\left\{y_1 \sin \left(\frac{1}{2}n\frac{2\pi}{m}\right) + y_2 \sin \left(\frac{3}{2}n\frac{2\pi}{m}\right)\right.$$
$$\left.+ y_3 \sin \left(\frac{5}{2}n\frac{2\pi}{m}\right) + \cdots + y_m \sin \left(\frac{2m-1}{2}\right)n\frac{2\pi}{m}\right\}$$

$$B_n = \frac{2}{m}\left\{y_1 \cos \left(\frac{1}{2}n\frac{2\pi}{m}\right) + y_2 \cos \left(\frac{3}{2}n\frac{2\pi}{m}\right)\right.$$
$$\left.+ y_3 \cos \left(\frac{5}{2}n\frac{2\pi}{m}\right) + \cdots + y_m \cos \left(\frac{2m-1}{2}\right)n\frac{2\pi}{m}\right\}$$

If the equation of the wave is desired in sine or cosine terms alone, the constant C and the angles θ or θ' for the sine or cosine series may be obtained by the method given on page 81. For any term such as the kth, the constant C and the angles are

$$C_k = \sqrt{A_k^2 + B_k^2}$$
$$\theta_k = \tan^{-1}\frac{B_k}{A_k}$$
$$\theta_k' = \tan^{-1}\frac{A_k}{B_k}$$

The precision of the constants and the angles is increased by making m large. For most commercial work, it is not necessary to carry the analysis beyond the seventh or the eleventh harmonic.

Theoretically, it is easy to calculate the constants by the method just outlined. The real difficulty lies in the amount of time involved. This may be reduced by arranging the work in a tabular form. The forms in use follow a method introduced by C. Runge,[1] in which the calculation of the constants is carried out by the aid of a systematically arranged schedule.

Fischer-Hinnen Method of Analyzing a Periodic Wave into the Components of Its Fourier Series.—The Fischer-Hinnen method,[2] which is also known as the *selected-ordinate method,* reduces to a minimum the work of determining the constants for any definite number of terms of the Fourier series of a wave. It is based on two mathematical laws which can easily be demonstrated.

Let the distance between any two points a and b, corresponding to a whole wave length, *i.e.*, corresponding to 360 degrees or 2π radians, be divided into m equal parts. Measure the length of the ordinate at the beginning of each part for any harmonic, such as the kth. Call the ordinates

$$Y_{m1},\ Y_{m2},\ Y_{m3} \text{ etc.}$$

where the first subscript m indicates the number of parts into which the wave is divided and the second subscript indicates the position of the ordinate.

If these ordinates are added,

$$\begin{aligned} \Sigma_1^m Y_m = A_k \sin k\alpha &+ A_k \sin\left\{k\frac{360^\circ}{m} + k\alpha\right\} \\ &+ A_k \sin\left\{2k\frac{360^\circ}{m} + k\alpha\right\} \\ &+ A_k \sin\left\{3k\frac{360^\circ}{m} + k\alpha\right\} \\ &\quad \cdots\cdots\cdots\cdots \\ &\quad \cdots\cdots\cdots\cdots \\ &+ A_k \sin\left\{(m-1)k\frac{360^\circ}{m} + k\alpha\right\} \qquad (26) \end{aligned}$$

[1] Elektrotechnische Zeitschrift, 1905, p. 247.

[2] Elektrotechnische Zeitschrift, vol. 22, p. 296, 1901.

The angle α is the angular distance, measured on the fundamental scale of angles, between the first ordinate and the point where the harmonic considered passes through zero increasing in a positive direction. A_k is the amplitude of the harmonic. (See Fig. 26.)

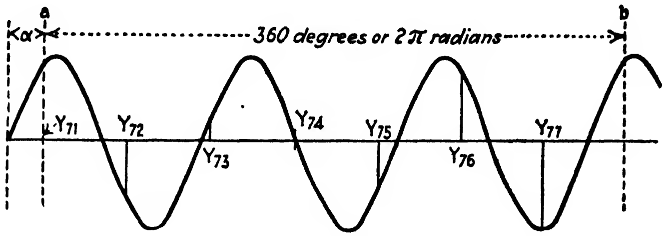


FIG. 26.

Expanding the angles in equation (26) gives

$$\begin{aligned}
\Sigma_1^m Y_m = {} & A_k \sin k\alpha \\
& + A_k \sin k \frac{360^\circ}{m} \cos k\alpha + A_k \cos k \frac{360^\circ}{m} \sin k\alpha \\
& + A_k \sin 2k \frac{360^\circ}{m} \cos k\alpha + A_k \cos 2k \frac{360^\circ}{m} \sin k\alpha \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + A_k \sin (m-1)\, k \frac{360^\circ}{m} \cos k\alpha \\
& \qquad + A_k \cos (m-1)k \frac{360^\circ}{m} \sin k\alpha \\
= {} & A_k \left\{ \sin k\alpha \left[1 + \cos k \frac{360^\circ}{m} + \cos 2k \frac{360^\circ}{m} + \right.\right. \\
& \qquad \left.\left. \cdots + \cos (m-1)k \frac{360^\circ}{m} \right] \right\} \\
& + A_k \left\{ \cos k\alpha \left[\sin k \frac{360^\circ}{m} + \sin 2k \frac{360^\circ}{m} + \right.\right. \\
& \qquad \left.\left. \cdots + \sin (m-1)k \frac{360^\circ}{m} \right] \right\} \qquad (27)
\end{aligned}$$

Since the sine of any whole number times 360 degrees is equal to zero and the cosine of any whole number times 360

degrees is unity, it is evident from inspection of equation (27) that, when $\frac{k}{m}$ is a whole number,

$$\Sigma_1^m Y_m = mA_k \sin k\alpha \tag{28}$$

The sum of m equally spaced ordinates for any harmonic, such as the kth, is, therefore, equal to m times the first ordinate when the order k of the harmonic is any whole number times the number of ordinates m.

If $\frac{k}{m}$ is not a whole number for the harmonic considered, the series represented by equation (27) reduces to zero. This can be shown by the aid of the two following trigonometrical formulas:

$$\cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos(m-1)\theta = -\frac{1}{2} + \frac{\cos(m-1)\theta - \cos m\theta}{2(1-\cos\theta)} \tag{29}$$

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin(m-1)\theta = \frac{\sin\left(\frac{m-1}{2}\right)\theta \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} \tag{30}$$

If θ is put equal to $\frac{k360°}{m}$ and $\frac{k}{m}$ is not a whole number, equation (29) reduces to -1 and equation (30) reduces to zero. Therefore, when $\frac{k}{m}$ is not a whole number,

$$\Sigma_1^m Y_m = A_k\,\{\sin k\alpha\,(1-1) + \cos k\alpha(0)\} = 0 \tag{31}$$

The relations given in equations (28) and (31) may be used to determine the components of the Fourier series for any periodic wave. Let Fig. 27 represent a fundamental and a third harmonic for a wave containing only a fundamental and a third harmonic.

Erect the two ordinates, a and b, 360 degrees apart, taking any convenient point such as a for the origin. Let the fundamental and the harmonic of the wave each be resolved into a sine and a cosine component referred to a as the origin from which to reckon time. Then the equation of the wave is

$$y = C_1 \sin(\omega t + \beta_1) + C_3 \sin 3(\omega t + \beta_3) \tag{32}$$
$$= A_1 \sin \omega t + B_1 \cos \omega t + A_3 \sin 3\omega t + B_3 \cos 3\omega t \tag{33}$$

where

$$\sqrt{A_1^2 + B_1^2} = C_1 \qquad \sqrt{A_3^2 + B_3^2} = C_3$$
$$\tan \beta_1 = \frac{B_1}{A_1} \qquad \tan \beta_3 = \frac{B_3}{A_3}$$

Divide the whole wave length between a and b into three equal parts by three equally spaced ordinates shown in Fig. 27. Call the lengths of these ordinates Y_{31}, Y_{32} and Y_{33}.

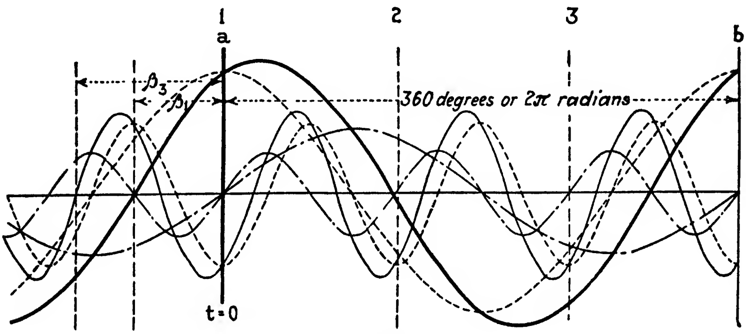


FIG. 27.

At the origin, $t = 0$ and each sine component of the wave [see equation (33)] is zero, but each cosine component has its maximum value. The length of the ordinate erected at the point $t = 0$ is, therefore, equal to the sum of the cosine terms of the fundamental and harmonics.

$$Y_{31} = B_1 + B_3$$

In general, if there are k harmonics, the ordinate erected at the point $t = 0$ is equal to the sum of the maximum values of the cosine terms for the fundamental and all harmonics.

$$Y_{m1} = B_1 + B_3 + B_5 + \cdots + B_k \tag{34}$$

Referring again to equation (33) and Fig. 27, it is obvious from equations (28) and (31) that if Y_{31}, Y_{32} and Y_{33} are the lengths of the ordinates at the points 1, 2 and 3,

$$\frac{1}{3}(Y_{31} + Y_{32} + Y_{33}) = B_3$$

or the maximum value of the cosine component for the third harmonic.

If the wave contained harmonics which were multiples of the third, one-third of the sum of the ordinates Y_{31}, Y_{32} and Y_{33} would be the sum of the maximum values of the cosine terms for the third and all harmonics which were multiples of the third. That is, in general,

$$\frac{1}{3}(Y_{31} + Y_{32} + Y_{33}) = (B_3 + B_9 + B_{15} + \text{etc.}) \qquad (35)$$

Similarly, if a wave containing harmonics is divided into five equal parts by five equally spaced ordinates, Y_{51}, Y_{52}, Y_{53}, Y_{54} and Y_{55}, the first ordinate being erected at the point a,

$$\frac{1}{5}(Y_{51} + Y_{52} + Y_{53} + Y_{54} + Y_{55}) = (B_5 + B_{15} + B_{25} + \text{etc.})$$

Dividing the wave into seven and into nine parts gives

$$\frac{1}{7}(Y_{71} + Y_{72} + \cdots + Y_{77}) = (B_7 + B_{21} + B_{35} + \text{etc}).$$

$$\frac{1}{9}(Y_{91} + Y_{92} + \cdots + Y_{99}) = (B_9 + B_{27} + B_{45} + \text{etc.})$$

In practice, it is convenient to erect the first ordinate at the point where the curve to be analyzed crosses the axis of time. In this case, Y_{m1} is zero and from equation (34)

$$B_1 + B_3 + B_5 + \text{etc.} = 0$$
$$B_1 = -B_3 - B_5 - \text{etc.}$$

For most current and voltage waves met in practice, harmonics of higher order than the seventh are not important. Relatively high harmonics may sometimes be produced by the armature teeth of motors or generators. If present in any appreciable magnitude, they are as a rule easily detected by inspection of the wave and may then be calculated.

If harmonics of higher order than the seventh are negligible,

$$B_1 = -B_3 - B_5 - B_7 \tag{36}$$

$$B_3 = \frac{1}{3}(Y_{31} + Y_{32} + Y_{33}) \tag{37}$$

$$B_5 = \frac{1}{5}(Y_{51} + Y_{52} + \cdots + Y_{55}) \tag{38}$$

$$B_7 = \frac{1}{7}(Y_{71} + Y_{72} + \cdots + Y_{77}) \tag{39}$$

If the ninth harmonic is present, as well as the third, fifth and seventh,

$$B_3 + B_9 = \frac{1}{3}(Y_{31} + Y_{32} + Y_{33}) \tag{40}$$

$$B_9 = \frac{1}{9}(Y_{91} + Y_{92} + \cdots + Y_{99}) \tag{41}$$

$$B_3 = \frac{1}{3}(Y_{31} + Y_{32} + Y_{33}) - B_9 \tag{42}$$

When the approximate equations [equations (36) to (39), inclusive] are used to analyze a wave, the base line may be appropriately divided to detect higher harmonics. If they exist in appreciable magnitudes, correction must be made for them by the method indicated for correcting the third for the presence of the ninth. [See equation (42).]

Each of the sine terms in the wave has its maximum value one-quarter of a period (measured on the scale of angles for the harmonic considered) from the initial ordinate Y_{m1}, or, in Fig. 27, page 103, from the ordinate Y_{31}. If the ordinates on Fig. 27 are shifted in the direction of lag (to the right) one-quarter of a period for the third harmonic, one-third of their sum is equal to the maximum value of the sine term for the third harmonic.

$$\frac{1}{3}(Y_{31}' + Y_{32}' + Y_{33}') = A_3 \tag{43}$$

The primes on the Y's indicate that they have been shifted in the direction of lag from their original position, *i.e.*, from the position in which they were drawn for determining the coefficients of the cosine terms, by one-quarter of a period for the harmonic considered.

If the wave contains harmonics which are multiples of the third,

$$\frac{1}{3}(Y_{31}' + Y_{32}' + Y_{33}') = (A_3 - A_9 + A_{15} - \text{etc.}) \qquad (44)$$

Similarly, if the Y_5 group of ordinates for the fifth harmonic is shifted in the direction of lag by one-quarter of a period for the fifth harmonic, *i.e.*, by $\frac{90}{5}$ degrees or $\frac{1}{5} \times \frac{\pi}{2}$ radians on the fundamental scale of angles,

$$\frac{1}{5}(Y_{51}' + Y_{52}' + \cdots + Y_{55}') = (A_5 - A_{15} + A_{25} - \text{etc.}) \qquad (45)$$

If the Y_7 group of ordinates is shifted in the direction of lag by one-quarter of a period for the seventh harmonic,

$$\frac{1}{7}(Y_{71}' + Y_{72}' + \cdots + Y_{77}') = (A_7 - A_{21} + A_{35} - \text{etc.}) \qquad (46)$$

If the harmonics above the seventh are negligible,

$$A_3 = \frac{1}{3}(Y_{31}' + Y_{32}' + Y_{33}') \qquad (47)$$

$$A_5 = \frac{1}{5}(Y_{51}' + Y_{52}' + \cdots + Y_{55}') \qquad (48)$$

$$A_7 = \frac{1}{7}(Y_{71}' + Y_{72}' + \cdots + Y_{77}') \qquad (49)$$

If the ninth harmonic is also present, A_3 as found by equation (47) must be corrected for that harmonic. For example,

$$A_9 = \frac{1}{9}(Y_{91}' + Y_{92}' + \cdots + Y_{99}') \qquad (50)$$

$$A_3 = \frac{1}{3}(Y_{31}' + Y_{32}' + Y_{33}') + A_9 \qquad (51)$$

If other high-order harmonics are present, the other coefficients, A_5 and A_7, must similarly be corrected. The method of correction is the same as indicated for the coefficients A_3 and B_3.

If the first ordinate, Y_{m1}, of the group of ordinates for the cosine series is shifted in the direction of lag one-quarter of a period for the fundamental, its length becomes

$$Y_{m1}' = A_1 - A_3 + A_5 - A_7 + A_9 - \text{etc.}$$
$$A_1 = Y_{m1}' - (-A_3 + A_5 - A_7 + A_9 - \text{etc.}) \qquad (52)$$

The even harmonics do not contribute to the ordinate Y_{m1}'

That equation (52) is true when either odd or even harmonics or both are present will be understood when the phase relations among the fundamental and harmonics of the sine terms are considered at the ordinate Y_{m1}'.

A change of α degrees in the position of any ordinate, measured on the scale of angles for the fundamental, is equivalent to a change in position of $k\alpha$ degrees with respect to the kth harmonic scale of angles. The following table is based on this relationship. This table gives the change in phase produced in each harmonic by moving the axis of reference in the direction of lag through one-quarter of a period for the fundamental.

Harmonic	Change in phase	Harmonic	Change in phase
Fundamental	$90°$	8th	$8 \times 90° = 720°$ $\Bumpeq 0°$
2d	$2 \times 90° = 180°$	9th	$9 \times 90° = 810°$ $\Bumpeq 90°$
3d	$3 \times 90° = 270°$	10th	$10 \times 90° = 900°$ $\Bumpeq 180°$
4th	$4 \times 90° = 360°$ $\Bumpeq 0°$	11th	$11 \times 90° = 990°$ $\Bumpeq 270°$
5th	$5 \times 90° = 450°$ $\Bumpeq 90°$	12th	$12 \times 90° = 1080°$ $\Bumpeq 0°$
6th	$6 \times 90° = 540°$ $\Bumpeq 180°$	13th	$13 \times 90° = 1170°$ $\Bumpeq 90°$
7th	$7 \times 90° = 630°$ $\Bumpeq 270°$	14th	$14 \times 90° = 1260°$ $\Bumpeq 180°$

The cosine terms for the fundamental and all harmonics are in phase at the ordinate Y_{m1} and add directly to it. The sine terms for the fundamental and all harmonics are zero at the ordinate Y_{m1} and contribute nothing to its magnitude. According to the table, the fifth, ninth, thirteenth etc. harmonics of the sine series have their maximum values at a point 90 degrees from the ordinate Y_{m1}, measured in the direction of lag on the fundamental scale of angles, and add directly to the maximum

value of the sine term of the fundamental at an ordinate Y_{m1}', erected 90 fundamental degrees in the direction of lag from the ordinate Y_{m1}. A shift in the position of the axis of 90 fundamental degrees in the direction of lag changes the phase of the third, seventh, eleventh etc. harmonics in the direction of lag by 270 degrees, measured on their own scales of angles, or by $360 - 270 = 90$ degrees in the direction of lead. They are, therefore, opposite in phase to the sine terms of the fundamental, the fifth, ninth, thirteenth etc. harmonics at the ordinate Y_{m1}' and subtract from it.

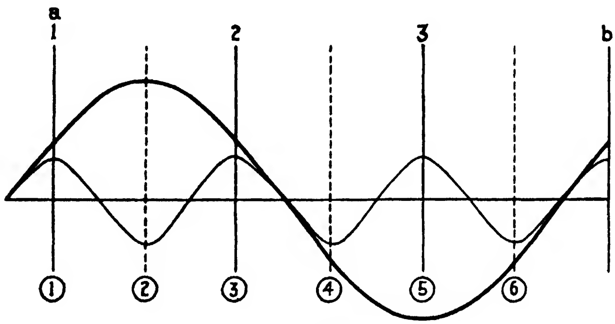


FIG. 28

The even harmonics of the sine terms are all changed in phase by either 180 or 0 degrees by a shift in the reference axis of 90 fundamental degrees in the direction of lag. They consequently all have zero values at the ordinate Y_{m1}', as well as at the ordinate Y_{m1}, and contribute nothing to it.

The cosine terms for the fundamental and all harmonics have zero values at the ordinate Y_{m1}' and contribute nothing to it.

Most current and voltage waves met in practice are symmetrical, *i.e.*, their positive and negative loops are identical except in sign. When the positive and negative loops of a wave are symmetrical, it is necessary to construct only the first loop, as the length of any given ordinate for the second loop may be determined from a suitably placed ordinate in the first loop.

Consider a third harmonic. Refer to Fig. 28.

Let the three ordinates marked 1, 2 and 3 at their upper ends be the ordinates for determining one component, such as the

cosine component, of the third harmonic. If there are no harmonics present which are multiples of the third,

$$B_3 = \frac{1}{3}(Y_{31} + Y_{32} + Y_{33})$$

Instead of dividing the whole wave into k parts (in this case three), where k is the order of the harmonic to be determined, let the half wave be divided into k parts, as shown in Fig. 28. The numbers for the k ordinates for the half wave or the $2k$ ordinates for the whole wave have circles around them and are placed at the bottom of the ordinates in Fig. 28.

The second of the six ordinates, *i.e.*, of the $2k$ ordinates, is obviously equal in magnitude but opposite in sign to the third of the three ordinates marked at their upper ends. Therefore,

$$B_3 = \frac{1}{3}(Y_{61} - Y_{62} + Y_{63}) \tag{53}$$

If, in addition to the third harmonic, the wave contains harmonics which are multiples of the third,

$$B_3 + B_9 + B_{15} + \text{etc.} = \frac{1}{3}(Y_{61} - Y_{62} + Y_{63}) \tag{54}$$

To determine the fifth harmonic, divide the half wave into five parts. This corresponds to dividing the whole wave into ten parts. Then

$$B_5 + B_{15} + B_{25} + \text{etc.} = \frac{1}{5}(Y_{10\,1} - Y_{10\,2} + Y_{10\,3} - Y_{10\,4} + Y_{10\,5}) \tag{55}$$

Similarly, by dividing the half wave into seven parts, the seventh harmonic plus the others of its group may be found. Dividing the half wave into seven parts corresponds to dividing the whole wave into fourteen parts,

$$B_7 + B_{21} + B_{35} + \text{etc.} = \frac{1}{7}(Y_{14\,1} - Y_{14\,2} + Y_{14\,3} - Y_{14\,4} + Y_{14\,5} - Y_{14\,6} + Y_{14\,7}) \tag{56}$$

When the loop of a *symmetrical* wave is divided into k parts by equally spaced ordinates, to determine the kth harmonic the

sum of the $\frac{k-1}{2}$ ordinates with even numbers is subtracted from the sum of the $\frac{k+1}{2}$ ordinates with odd numbers.

It is convenient, when analyzing a wave containing only odd harmonics, to divide the half wave into $2k$ equal parts by $2k$ equally spaced ordinates, erecting the first ordinate where the wave crosses the axis of time. When $2k$ ordinates are thus drawn in the half wave, those with odd numbers are used for determining the coefficients of the cosine terms. Those with even numbers are one-quarter period (for the kth harmonic) from the others and are used for determining the coefficients of the sine terms. For example, suppose a wave contains a fifth harmonic but no others, such as the fifteenth, which would be included with the fifth. Let the half wave be divided into ten parts and number the ordinates $Y_1, Y_2, Y_3, \ldots, Y_{10}$. Then

$$B_5 = \frac{1}{5}(Y_1 - Y_3 + Y_5 - Y_7 + Y_9) \tag{57}$$

$$A_5 = \frac{1}{5}(Y_2 - Y_4 + Y_6 - Y_8 + Y_{10}) \tag{58}$$

That is, the ordinates with odd numbers are used with alternate signs to determine the B coefficients, and the ordinates with even numbers are used with alternate signs to determine the A coefficients. In each case the first ordinate is considered positive.

An analysis of a wave containing only odd harmonics will make this clear.

When calculating the B and A coefficients of a wave by the Fischer-Hinnen method, it must be remembered that the coefficients are not obtained separately but in groups. For this reason, it is always necessary to determine, by an approximate division of the wave, the highest order harmonic that is present in appreciable magnitude, in order that the coefficients given by the approximate equations (36) to (39) and (47) to (49), inclusive, may be corrected for the presence of high-order harmonics.

Example of the Analysis of a Wave Containing only Odd Harmonics by the Fischer-Hinnen Method.—The wave of a 60-cycle current will be analyzed. Analysis shows that this wave contains a pronounced third harmonic, a fairly large fifth

harmonic and small seventh and ninth harmonics. The wave is symmetrical. There are, therefore, no even harmonics. The harmonics above the ninth are negligible.

The half wave, with the proper ordinates for determining the coefficients for the third harmonic, is shown in Fig. 29. Since the ninth harmonic is not negligible, these coefficients must be corrected for its presence.

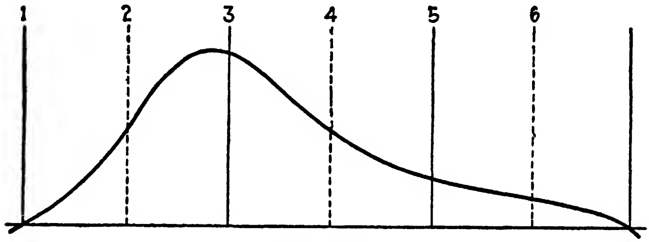


FIG. 29.

Measurements made on the original curve from which Fig. 29 was reproduced give the following lengths of the ordinates in inches:

$Y_1 = 0.00$	$Y_2 = 0.99$	$Y_3 = 1.80$	$Y_4 = 1.01$
$Y_5 = 0.51$	$Y_6 = 0.30$		
0.51	1.29	1.80	1.01

The scale for the ordinates of the original curve is 1 ampere per inch.

$$
\begin{aligned}
B_3 + B_9 &= \frac{1}{3}(Y_1 - Y_3 + Y_5) \\
&= \frac{1}{3}(0.51 - 1.80) \\
&= -0.43 \text{ inch} \\
&= -0.43 \times 1 = -0.43 \text{ ampere} \\
A_3 - A_9 &= \frac{1}{3}(Y_2 - Y_4 + Y_6) \\
&= \frac{1}{3}(1.29 - 1.01) \\
&= 0.093 \text{ inch} \\
&= 0.093 \times 1 = 0.093 \text{ ampere}
\end{aligned}
$$

The half wave with the proper ordinates for finding the fifth harmonic is shown in Fig. 30.

Measurements made on the curve from which Fig. 30 was reproduced give the following lengths of the ordinates in inches:

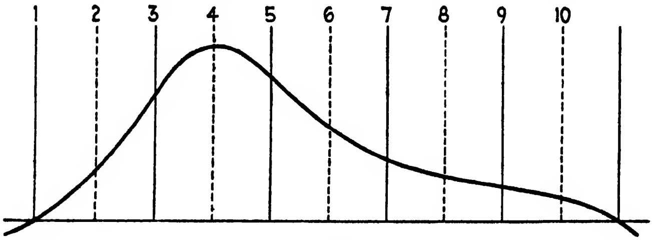


FIG. 30.

$Y_1 = 0.00$	$Y_2 = 0.51$	$Y_3 = 1.32$	$Y_4 = 1.82$
$Y_5 = 1.52$	$Y_6 = 1.01$	$Y_7 = 0.65$	$Y_8 = 0.44$
$Y_9 = 0.33$	$Y_{10} = 0.22$		
1.85	1.74	1.97	2.26

$$
\begin{aligned}
B_5 &= \frac{1}{5}(Y_1 - Y_3 + Y_5 - Y_7 + Y_9) \\
&= \frac{1}{5}(1.85 - 1.97) \\
&= -0.024 \text{ inch} \\
&= -0.024 \times 1 = -0.024 \text{ ampere} \\
A_5 &= \frac{1}{5}(Y_2 - Y_4 + Y_6 - Y_8 + Y_{10}) \\
&= \frac{1}{5}(1.74 - 2.26) \\
&= -0.10 \text{ inch} \\
&= -0.10 \times 1 = -0.10 \text{ ampere}
\end{aligned}
$$

Further analysis shows that

$$
\begin{aligned}
B_7 &= 0.027 \text{ ampere} \\
A_7 &= 0.003 \text{ ampere} \\
B_9 &= 0.004 \text{ ampere} \\
A_9 &= 0.014 \text{ ampere}
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= (B_3 + B_9) - B_9 \\
&= -0.43 - 0.004 = -0.43 \text{ ampere} \\
A_3 &= (A_3 - A_9) + A_9 \\
&= 0.093 + 0.014 = 0.107 \text{ ampere}
\end{aligned}
$$

From equation (36), page 105,

$$\begin{aligned} B_1 &= -(B_3 + B_5 + B_7 + B_9) \\ &= -\{(-0.43) + (-0.024) + (0.027) + (0.004)\} \\ &= 0.42 \text{ ampere} \end{aligned}$$

From equation (52), page 107,

$$A_1 = Y_{m1}' - (-A_3 + A_5 - A_7 + A_9)$$

By measurement on the original figure, the ordinate 4 (see Fig. 29), which is displaced 90 fundamental degrees in the direction of lag from the ordinate marked 1, is equal to 1.01 amperes.

$$\begin{aligned} A_1 &= 1.01 - \{-(0.107) + (-0.10) - (0.003) + (0.014)\} \\ &= 1.21 \text{ amperes} \end{aligned}$$

$$\begin{aligned} i &= A_1 \sin \omega t + B_1 \cos \omega t \\ &\quad + A_3 \sin 3\omega t + B_3 \cos 3\omega t \\ &\quad + A_5 \sin 5\omega t + B_5 \cos 5\omega t \\ &\quad + A_7 \sin 7\omega t + B_7 \cos 7\omega t \\ &\quad + A_9 \sin 9\omega t + B_9 \cos 9\omega t \\ &= 1.21 \sin 377t + 0.42 \cos 377t \\ &\quad + 0.107 \sin 1131t - 0.43 \cos 1131t \\ &\quad - 0.10 \sin 1885t - 0.024 \cos 1885t \\ &\quad + 0.003 \sin 2639t + 0.027 \cos 2639t \\ &\quad + 0.014 \sin 3393t + 0.004 \cos 3393t \end{aligned}$$

$$\begin{aligned} C_1 &= \sqrt{A_1^2 + B_1^2} = \sqrt{(1.21)^2 + (0.42)^2} \\ &= 1.28 \text{ amperes} \end{aligned}$$

$$\tan \theta_1 = \frac{B_1}{A_1} = \frac{0.42}{1.21} = 0.347$$

$$\theta_1 = +19.1 \text{ degrees}$$

$$\begin{aligned} C_3 &= \sqrt{A_3^2 + B_3^2} = \sqrt{(0.107)^2 + (-0.43)^2} \\ &= 0.443 \text{ ampere} \end{aligned}$$

$$\tan \theta_3 = \frac{B_3}{A_3} = \frac{-0.43}{0.107} = -4.02$$

$$\theta_3 = -76.0 \text{ degrees}$$

$$\begin{aligned} C_5 &= \sqrt{A_5^2 + B_5^2} = \sqrt{(-0.10)^2 + (-0.024)^2} \\ &= 0.103 \text{ ampere} \end{aligned}$$

$$\tan \theta_5 = \frac{B_5}{A_5} = \frac{-0.024}{-0.10} = 0.24$$

$$\theta_5 = -166.5 \text{ degrees}$$

$$C_7 = \sqrt{A_7^2 + B_7^2} = \sqrt{(0.003)^2 + (0.027)^2}$$
$$= 0.027 \text{ ampere}$$
$$\tan \theta_7 = \frac{B_7}{A_7} = \frac{0.027}{0.003} = 9.0$$
$$\theta_7 = +84 \text{ degrees}$$
$$C_9 = \sqrt{A_9^2 + B_9^2} = \sqrt{(0.014)^2 + (0.004)^2}$$
$$= 0.0146 \text{ ampere}$$
$$\tan \theta_9 = \frac{B_9}{A_9} = \frac{0.004}{0.014} = 0.286$$
$$\theta_9 = +16 \text{ degrees}$$

$$i = 1.28 \sin (377t + 19.^\circ1) + 0.443 \sin (1131t - 76.^\circ0) + 0.103 \sin (1885t - 166.^\circ5) + 0.027 \sin (2639t + 84^\circ) + 0.015 \sin (3393t + 16^\circ)$$

Form Factor.—The form factor for a symmetrical alternating current or voltage is the ratio of its effective or root-mean-square value to its average value.

$$\text{Form factor} = \frac{\text{effective value}}{\text{average value}} = \frac{\sqrt{\frac{1}{T}\int_0^T e^2 dt}}{\frac{2}{T}\int_0^{\frac{T}{2}} e dt} \tag{59}$$

For a sinusoidal wave the form factor is

$$\frac{\frac{E_m}{\sqrt{2}}}{\frac{2E_m}{\pi}} = 1.11$$

The form factor is usually less than 1.11 for a flat-topped wave, *i.e.*, a wave that is flatter than a sinusoidal wave. It is usually greater than 1.11 for a peaked wave, *i.e.*, one that is more peaked than a sinusoidal wave.

The form factor is generally of no importance in comparing two waves, as two waves having totally different wave forms may have equal form factors. For example, the following wave has the same form factor as a sinusoidal wave, but it is of totally different wave shape:

$$i = I_{m1} \sin \omega t + \frac{3}{4} I_{m1} \sin 3\omega t$$

This wave and a sine wave having the same root-mean-square or effective value are plotted in Fig. 31. There are, however, a few conditions under which form factor is of importance.

Amplitude, Crest or Peak Factor.—The amplitude, crest or peak factor of an alternating wave is the ratio of its maximum value to its root-mean-square or effective value. A knowledge of the amplitude or peak factor is of importance in testing insulation, since the stress to which an insulation is subjected by a given impressed voltage depends upon the maximum value of the voltage and not upon its root-mean-square or effective value. A knowledge of the form factor of a voltage in connection with

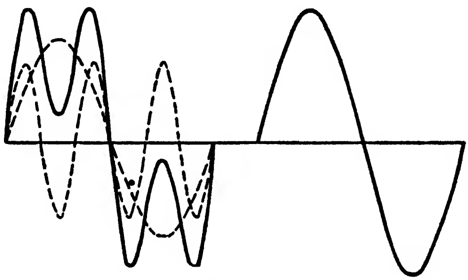


Fig. 31.

insulation testing. however, is of little if any value. The peak factor of a sinusoidal wave is $\sqrt{2} = 1.41$.

Deviation Factor.—The deviation factor of a wave is the ratio of the maximum difference between the corresponding ordinates of the actual wave and an equivalent sine wave of equal length, to the maximum ordinate of the equivalent sine wave, when the waves are superposed in such a way as to make the maximum difference between corresponding ordinates as small as possible. (For definition of equivalent sine wave see the next paragraph.) Except in special cases, a deviation factor of 0.10 is permissible in the wave form of commercial electrical machinery.

Equivalent Sine Waves.—In most of the alternating-current phenomena which are met in practice, neither the voltage nor the current is sinusoidal, although both are periodic. In many cases where the wave forms do not differ greatly from a sinusoid. it is sufficiently accurate to replace the non-sinusoidal waves of voltage and current by equivalent sine waves. These equivalent sine

waves have the same root-mean-square or effective values as the actual waves they replace, and their phase is so chosen as to make $EI \cos \theta$ for the equivalent sine waves equal to the actual power. The sign of the phase angle between the equivalent sine waves is made the same as that of the phase angle between the fundamentals of the actual waves. When the fundamentals are in phase, but the power factor is not unity, the sign of the equivalent phase angle is indeterminate.

Complex waves, which differ much in wave form, cannot be replaced by their equivalent sine waves, when they are to be added or subtracted, without danger of introducing considerable error. (See page 119.)

Equivalent Phase Difference.—The angle θ, determined by $\theta = \cos^{-1} \dfrac{P}{EI}$ when voltage and current are not sinusoidal waves, is known as the *equivalent phase angle.* I and E are the effective or root-mean-square values of the non-sinusoidal waves and P is the average power. The equivalent phase difference is the phase angle that must be used with the equivalent sine waves of current and voltage to produce the true power. The equivalent phase difference may often be misleading, since the presence of harmonics in one wave which are not present in the other lowers the power factor even if there is no displacement between the waves. In general, θ, as determined by $\theta = \cos^{-1} \dfrac{P}{EI}$, has no real physical significance except when sine waves are considered.

Example of Equivalent Sine Waves.—The analysis of the voltage and current waves for a certain circuit has shown them to be of the forms

$$e = 100 \sin \omega t + 20 \sin (3\omega t + 60°) + 15 \sin (5\omega t - 40°)$$
$$i = 40 \sin (\omega t - 30°) + 5 \sin (3\omega t + 20°)$$
$$E = \sqrt{\frac{(100)^2 + (20)^2 + (15)^2}{2}} = 72.9 \text{ volts}$$
$$I = \sqrt{\frac{(40)^2 + (5)^2}{2}} = 28.5 \text{ amperes}$$
$$P = \frac{100 \times 40}{2} \cos 30° + \frac{20 \times 5}{2} \cos 40°$$
$$= 1770 \text{ watts}$$

$$\text{Power factor} = \frac{1770}{72.9 \times 28.5} = 0.852$$

$$\text{Equivalent phase difference} = \theta = \cos^{-1} 0.852$$
$$= 31.6 \text{ degrees}$$

If the equivalent sine *voltage* is taken zero when time t is zero, the equivalent sine waves of voltage and current are

$$e = \sqrt{2} \times 72.9 \sin \omega t$$
$$= 103.1 \sin \omega t$$
$$i = \sqrt{2} \times 28.5 \sin (\omega t - 31.^\circ 6)$$
$$= 40.3 \sin (\omega t - 31.^\circ 6)$$

If the equivalent sine *current* is taken zero when time t is zero, the equivalent sine waves are

$$e = 103.1 \sin (\omega t + 31.^\circ 6)$$
$$i = 40.3 \sin \omega t$$

It should be noticed that the fifth harmonic in the voltage contributes nothing to the power, since there is no component in the current of the same frequency. Although it contributes nothing to power, it does increase the voltage and, therefore, lowers the power factor. Because the fifth harmonic contributes nothing to power, it cannot be neglected in finding the root-mean-square or effective value of the voltage.

As has already been stated, the power factor of a circuit cannot be unity unless the current and voltage contain like harmonics, and then the relative magnitudes and the phase relations of the harmonics must be identical in the two waves. The power factor of a circuit containing nothing but pure resistance cannot be unity unless the temperature coefficient of the material of which the resistance is made is zero or the dimensions of the resistance unit are such that there is no appreciable change in its resistance during a cycle. If a resistance unit is made of fine wire of high-temperature coefficient, its resistance changes appreciably during a cycle. If a sinusoidal voltage is impressed on such a resistance, the current is flatter than a sinusoid and, therefore, contains harmonics, among which is a marked third. The power factor of a circuit whose resistance

varies with current during a cycle cannot be unity, even if the circuit contains nothing but pure resistance. For commercial circuits, the change of the resistance with current during a cycle is usually too small to produce any noticeable effect on the shape of the current wave.

Consider a case where there is no phase displacement between the current and voltage waves, but the current wave contains a harmonic which is not present in the voltage. Let

$$e = 100 \sin 2\pi 60t$$
$$i = 10 \sin 2\pi 60t + 5 \sin 6\pi 60t$$

The voltage is a pure sinusoidal wave, while the current contains a fifty per cent third harmonic. This third harmonic contributes nothing to the power, since the voltage has no third harmonic. The root-mean-square or effective value of the voltage and the current are, respectively,

$$E = \sqrt{\frac{(100)^2}{2}} = 70.7 \text{ volts}$$
$$I = \sqrt{\frac{(10)^2 + (5)^2}{2}} = 7.91 \text{ amperes}$$
$$P = \frac{100 \times 10}{2} \cos (0° - 0°) + 0$$
$$= 500 \text{ watts}$$
$$\text{Power factor} = \frac{500}{70.7 \times 7.91} = 0.895$$

Equivalent phase difference $= \theta = \cos^{-1} 0.895 = \pm 26.5$ degrees

The equivalent sine waves are

$$e = 100 \sin 2\pi 60t$$
$$i = \sqrt{2} \times 7.91 \sin (2\pi 60t \pm 26.^\circ 5)$$
$$= 11.18 \sin (2\pi 60t \pm 26.^\circ 5)$$

Although the power in the example just given is due entirely to the sinusoidal voltage, $e = 100 \sin 2\pi 60t$, and to the sinusoidal current, $i = 10 \sin 2\pi 60t$, the latter is not the equivalent sine wave of current since it does not have the proper ampere value or phase relation.

Where an exact analysis of any particular problem is essential, the substitution of the equivalent sine waves for the actual voltage and current is not permissible.

Addition and Subtraction of Non-sinusoidal Waves.—When non-sinusoidal currents or voltages are to be added or subtracted, each must first be expressed in terms of its Fourier series, *i.e.*, in terms of its fundamental and harmonics. The fundamentals and the harmonics of like frequency may then be added or subtracted vectorially to give the fundamental and the harmonics of the resultant wave. Equivalent sine waves cannot be added or subtracted vectorially except when the wave forms are identical and the phase displacement is zero. If the wave forms are very different or the phase displacement between the waves is great, the error produced by adding or subtracting the equivalent sine waves may be large.

Example of Addition of Non-sinusoidal Waves.—Let the following voltage be impressed on a circuit having two branches in parallel, one consisting of a resistance of 5 ohms in series with a condenser of 132.7 microfarads capacitance, the other consisting of 10 ohms resistance in series with an inductance of 0.0398 henry.

$$e = 200 \sin (377t + 10°) + 75 \sin (1131t + 30°) + 50 \sin (1885t + 50°)$$

The currents in the two branches are (see Chapter VII)

$$i' = 9.71 \sin (377t + 85.^\circ96) + 9.00 \sin (1131t + 83.^\circ15) + 7.81 \sin (1885t + 88.^\circ66)$$

$$i'' = 11.09 \sin (377t - 46.^\circ31) + 1.625 \sin (1131t - 47.^\circ47) + 0.661 \sin (1885t - 32.^\circ40)$$

Capital letters with the subscripts 1, 3 and 5 represent root-mean-square values of the fundamental and harmonics. The subscript 0 used with the subscripts 1, 3 and 5 indicates resultants.

Consider the vectors representing the fundamentals of the currents at the instant when time t is zero. Their vector expressions are

$$\sqrt{2}\,\bar{I}_1' = 9.71(\cos 85°96 + j \sin 85°96)$$
$$= 0.684 + j9.68$$
$$\sqrt{2}\,\bar{I}_1'' = 11.09(\cos 46°31 - j \sin 46°31)$$
$$= 7.67 - j8.03$$
$$\sqrt{2}\,\bar{I}_{01} = 8.35 + j1.65$$
$$\sqrt{2}\,\bar{I}_{01} = \sqrt{(8.35)^2 + (1.65)^2} = 8.51 \text{ amperes maximum}$$
$$\tan\theta_{01} = \frac{1.65}{8.35} = 0.1976 \qquad \theta_{01} = 11.18 \text{ degrees}$$

Consider the vectors representing the third harmonics at the instant when t is zero.

$$\sqrt{2}\,\bar{I}_3' = 9.00(\cos 83°15 + j \sin 83°15)$$
$$= 1.074 + j8.94$$
$$\sqrt{2}\,\bar{I}_3'' = 1.625(\cos 47°47 - j \sin 47°47)$$
$$= 1.098 - j1.198$$
$$\sqrt{2}\,\bar{I}_{03} = 2.172 + j7.74$$
$$\sqrt{2}\,\bar{I}_{03} = \sqrt{(2.172)^2 + (7.74)^2} = 8.04 \text{ amperes maximum}$$
$$\tan\theta_{03} = \frac{7.74}{2.172} = 3.563 \qquad \theta_{03} = 74.32 \text{ degrees}$$

Consider the vectors representing the fifth harmonics at the instant t is zero.

$$\sqrt{2}\,\bar{I}_5' = 7.81(\cos 88°66 + j \sin 88°66)$$
$$= 0.183 + j7.81$$
$$\sqrt{2}\,\bar{I}_5'' = 0.661(\cos 32°40 - j \sin 32°40)$$
$$= 0.558 - j0.354$$
$$\sqrt{2}\,\bar{I}_{05} = 0.741 + j7.46$$
$$\sqrt{2}\,\bar{I}_{05} = \sqrt{(0.741)^2 + (7.46)^2} = 7.50 \text{ amperes maximum}$$
$$\tan\theta_{05} = \frac{7.46}{0.741} = 10.06 \qquad \theta_{05} = 84.32 \text{ degrees}$$

The resultant current wave is

$$i_0 = 8.51 \sin(377t + 11°18) + 8.04 \sin(1131t + 74°32) + 7.50 \sin(1885t + 84°32)$$

$$I_0 = \sqrt{\frac{(8.51)^2 + (8.04)^2 + (7.50)^2}{2}} = 9.84 \text{ amperes effective}$$

The resultant power is

$$P_0 = \frac{200 \times 8.51}{2} \cos (10^\circ - 11^\circ\!.18) + \frac{75 \times 8.04}{2} \cos (30^\circ - 74^\circ\!.32) + \frac{50 \times 7.50}{2} \cos (50^\circ - 84^\circ\!.32)$$

$$= 851 + 216 + 155 = 1222 \text{ watts}$$

The power in each branch of the parallel circuit is

$$P' = \frac{200 \times 9.71}{2} \cos (-75^\circ 96) + \frac{75 \times 9.00}{2} \cos (-53^\circ\!.15) + \frac{50 \times 7.81}{2} \cos (-38^\circ\!.66)$$

$$= 235.4 + 202.3 + 152.4 = 590.1 \text{ watts}$$

$$P'' = \frac{200 \times 11.09}{2} \cos (+56^\circ\!.31) + \frac{75 \times 1.625}{2} \cos (+77^\circ\!.47) + \frac{50 \times 0.661}{2} \cos (+82^\circ\!.40)$$

$$= 615.2 + 13.2 + 2.2 = 630.6 \text{ watts}$$

$$P_0 = P' + P'' = 590 + 631 = 1221 \text{ watts}$$

The maximum value of the equivalent sine wave of voltage is

$$\sqrt{2}E = \sqrt{(200)^2 + (75)^2 + (50)^2} = 219.3 \text{ volts maximum}$$

The maximum values of the equivalent sine waves of the currents are

$$\sqrt{2}\, I' = \sqrt{(9.71)^2 + (9.00)^2 + (7.81)^2} = 15.37 \text{ amperes}$$
maximum

$$\sqrt{2}\, I'' = \sqrt{(11.09)^2 + (1.625)^2 + (0.661)^2} = 11.23 \text{ amperes}$$
maximum

$$(\text{Power factor})' = \frac{590.1}{\dfrac{219.3}{\sqrt{2}} \times \dfrac{15.37}{\sqrt{2}}} = 0.3501$$

Equivalent phase difference $= \theta' = \cos^{-1} 0.3501 = 69.50$ degrees

$$(\text{Power factor})'' = \frac{630.6}{\dfrac{219.3}{\sqrt{2}} \times \dfrac{11.23}{\sqrt{2}}} = 0.5121$$

Equivalent phase difference $= \theta'' = \cos^{-1} 0.5121 = 59.20$ degrees

If the equivalent sine voltage is taken zero when t is zero, the equivalent sine waves are

$$\begin{aligned} e &= 219.3 \sin 377t \\ i' &= 15.37 \sin (377t + 69^\circ.50) \\ i'' &= 11.23 \sin (377t - 59^\circ.20) \end{aligned}$$

Add the equivalent sine currents as if they were actually sinusoidal waves. Consider the vectors representing them at the instant t is zero.

$$\begin{aligned} \sqrt{2}\, \bar{I}' &= 15.37 (\cos 69^\circ.50 + j \sin 69^\circ.50) \\ &= 5.38 + j14.4 \\ \sqrt{2}\, \bar{I}'' &= 11.23 (\cos 59^\circ.20 - j \sin 59^\circ.20) \\ &= 5.73 - j9.64 \\ \sqrt{2}\, \bar{I}_0 &= 11.1\dot{1} + j4.76 \\ \sqrt{2}\, I_0 &= \sqrt{(11.11)^2 + (4.76)^2} = 12.09 \text{ amperes maximum} \\ I_0 &= \frac{12.09}{\sqrt{2}} = 8.56 \text{ amperes effective} \end{aligned}$$

Adding the equivalent sine waves vectorially, in the example just given, gives 8.56 amperes for the resultant current instead of 9.84 amperes, the correct value. The error is 13 per cent. If both branches of the divided circuit had contained similar constants, the error of adding the equivalent sine waves would probably have been less. It could not have been zero unless the wave forms were identical and the two waves were in phase.

For example, suppose the wave forms of the two currents had been identical and each had contained a third harmonic. A phase displacement of 60 degrees between the waves, *i.e.*, between their fundamentals, would have made the third harmonic in the resultant zero. The effect of the third harmonics would not have canceled if the equivalent sine waves had been added.

It is readily seen from what precedes that whenever the voltage, current and power in the component parts of a circuit with parallel branches are measured, and the equivalent sine waves of current determined from the instrument readings are added vectorially as if they were really sinusoidal waves, the resultant current thus determined may be in considerable error if the wave forms of the component currents differ greatly from sinusoids. A similar statement holds regarding the addition of the equivalent sine waves of voltage drop across the component parts of a series circuit.

CHAPTER V

CIRCUITS CONTAINING RESISTANCE, INDUCTANCE AND CAPACITANCE

Coefficient of Self-induction or the Self-inductance of a Circuit.—In the neighborhood of an electric circuit carrying a current, there exists a magnetic field whose intensity at any point is dependent upon the strength of the current, the configuration of the circuit and the distance of the point from the circuit. If the current alters its value, the field is also altered, increasing with increase of current and decreasing with decrease of current. This magnetic field is a definite seat of energy, and for its production requires, therefore, a definite expenditure of energy, determined in amount by the flux and the conducting ampere turns of the circuit with which this flux is linked.

The linkages of flux with turns constitute one of the most important factors of any circuit. The change in the number of linkages per unit current for an electric circuit is called the *coefficient of self-induction* or the *self-inductance.* The self-inductance of the circuit is denoted by the symbol L.

If $d\varphi$ is the change in flux linking a circuit of N turns produced by a change di in the current, the coefficient of self-induction or the self-inductance of the circuit is

$$L = N\frac{d\varphi}{di}$$

It is the rate of change of flux linkages of a circuit with respect to the current it carries. If the flux linkages per unit current are constant, the coefficient of self-induction may be written

$$L = N\frac{\varphi}{I}$$

where φ is the flux, produced by the current I, which links with the N turns. In general, all the flux does not link with all the

turns of the circuit. In such cases the calculation of the coefficient of self-induction becomes more or less difficult. In most cases accurate calculation is impossible.

Coefficient of Mutual Induction or the Mutual Inductance of a Circuit.—When two circuits are so related that a change in the current in one produces a change in the flux linking the other, the circuits are said to possess mutual inductance. The mutual inductance or coefficient of mutual induction of a circuit with respect to another circuit is the change in the flux linkages of the second produced by a change of one unit of current in the first. In other words, it is the rate of change of flux linkages of the second circuit with respect to the current in the first. Mutual inductance will be considered more in detail later.

Henry.—When the number of flux linkages due to one abampere flowing in a circuit is 10^9, the circuit is said to possess a self-inductance of one henry. This definition holds only when flux is proportional to current; in other words, it is strictly true only when there is no magnetic material present. When there is no magnetic material present, self-inductance is constant. When flux is not proportional to current, a circuit is said to have a self-inductance of one henry when the rate of change of flux linkage with respect to current in abamperes is 10^9.

Energy of the Field.—If the current in a circuit remains constant in value, there is no expenditure of energy in maintaining the field. This excludes the energy dissipated in heat in the electric circuit itself due to the I^2r loss. If, however, the field increases, there is a reaction developed which requires an expenditure of electrical energy by the circuit to overcome it. This electrical energy appears as the magnetic energy of the increased field. If, on the other hand, the field diminishes, there is a reaction in the opposite direction and, in virtue of this, energy is contributed by the magnetic field to the electric circuit. The reaction in each case takes the form of an electromotive force, the magnitude of which depends on the time rate of change of flux linkages. This electromotive force is known as the *electromotive force of self-induction.* Expressed as a rise in electromotive force it is

$$e = -\frac{d}{dt}(N\varphi)$$

where e is the electromotive force of self-induction, N the number of conducting turns in the circuit and φ the flux linked with the turns N. $N\varphi$ is the number of flux linkages of the circuit.

Effect of Self-inductance for a Circuit Carrying an Alternating Current.—It is evident that if a circuit carries an alternating current such, for example, as a simple harmonic current, there is an alternate increase and diminution in the energy of the magnetic field and this gives rise to a reactive electromotive force. If a complete period for the current is considered, it is found that, during one-half of this period, energy is supplied by the circuit to the magnetic field and, during the other half of the period, energy is supplied by the field to the circuit. When the current is increasing in either a positive or a negative direction, the establishment of energy in the field sets up an opposing electromotive force which retards the flow of current, thus decreasing its rate of increase. This results in the current reaching a given value later than it would have reached it provided there were no such opposing electromotive force. While the current is decreasing, the field contributes energy to the circuit and diminishes the rate at which the current falls, thus causing it to pass through a given value later than it would have done, if no energy were returned to the circuit by the field. The net effect is to decrease the maximum positive and negative values reached by the current during a cycle and to cause the current to lag behind the impressed voltage producing it. In the flow, therefore, of a simple harmonic current in a circuit which possesses self-inductance, the value of the current is less than if there were no self-inductance and the current lags by a certain angle with respect to the impressed voltage.

Inductance and resistance are very different in their effects. Inductance opposes only a change in the current and is like mass in mechanics. Resistance opposes the flow of a steady current as well as a variable current, and is analogous to friction in mechanics. The kinetic energy of a moving mass is $\frac{1}{2}MV^2$, where M is the mass and V its velocity. For the electric circuit containing self-inductance, the kinetic energy is $\frac{1}{2}LI^2$, where L is the self-inductance and I the current. For the circuit, the kinetic energy is the energy in the magnetic field set up by the current.

Capacitance.—The capacitance of a condenser is measured by the charge required to raise its potential by unity. It is equal to the ratio of charge to voltage.

$$C = \frac{Q}{V}$$

A condenser is said to have a capacitance of one farad when a charge of one coulomb raises it to a potential of one volt. This unit is too large for practical use. For this reason, the capacitance of condensers is ordinarily expressed in microfarads. The resistance to direct current of a well-made condenser is very high, and for most practical purposes may be considered infinite. This does not mean, however, that a condenser connected across an alternating-current circuit takes no current. It alternately charges and discharges at the frequency of the circuit in which it is connected, and, therefore, takes an alternating current of perfectly definite effective or root-mean-square value. The resistance of a condenser to an alternating current is equal to the average power it takes, when placed across an alternating-current circuit, divided by the square of the effective or root-mean-square current taken from the mains. The power absorbed is the I^2r loss in the condenser caused by the alternating charging current, plus the hysteresis loss in the dielectric. The latter loss is caused by the varying stresses produced in the dielectric by the alternating voltage impressed across the condenser terminals. The dielectric hysteresis loss per unit volume depends on the nature of the dielectric, the maximum potential gradient to which it is subjected and the frequency. The dielectric hysteresis loss for air is zero. It is small in most dielectrics at commercial frequencies, *i.e.*, at frequencies of 60 cycles or lower. At frequencies used for the generation, transmission and utilization of power, the losses in commercial condensers are small and for most purposes may be neglected.

Effect of Capacitance in an Alternating-current Circuit.—The effect of capacitance in a circuit, so far as the phase relation between current and voltage is concerned, is just the opposite of that of inductance. Inductance causes the current in a circuit to lag the voltage drop across its terminals. Capacitance

causes the current in a circuit to lead the voltage drop across its terminals.

The voltage drop across the terminals of a condenser is

$$e = \frac{q}{C}$$

where C is the capacitance of the condenser and q its charge. When the charge on the condenser is a maximum, the voltage drop across its terminals is also a maximum. When the charge is zero, the voltage drop is also zero. The current taken by a condenser without leakance at any instant is equal to the difference between the voltage impressed across its terminals and the opposing electromotive force, due to the condenser charge divided by the condenser resistance. The opposing electromotive force is $e = \frac{q}{C}$. Therefore, when the charge is a maximum the current must be a minimum and when the charge is a minimum the current must be a maximum. There can be no charge in the condenser until there has been current flow, since $q = \int idt$. The current must consequently lead the charge and, since the voltage drop across the terminals of a condenser is $e = \frac{q}{C}$, it must also lead the voltage drop.

The effect of a condenser in an electric circuit is similar to the effect of elasticity in a mechanical system. As a condenser charges, its back electromotive force rises with the charge and offers an increasing opposition to further current flow, the opposition increasing in proportion to the charge. This back electromotive force is similar in its effect on current flow to the reaction of a spring which is being stretched. The reaction of the spring offers increasing opposition to further stretching.

Circuit Containing Constant Resistance and Constant Self-inductance in Series.—In any energy relation whatsoever, the sum of the actions and the reactions must be zero. The action for the ordinary electric circuit is balanced by the impressed voltage drop. The reactions for a circuit containing resistance and self-inductance in series are the ohmic and reactive drops, the

latter being due to the presence of self-induction. The condition of the circuit is completely determined by the equation

$$e = ri + N\frac{d\varphi}{dt} = ri + L\frac{di}{dt} \tag{1}$$

In general, this can be solved only when φ is proportional to i, *i.e.*, when L is constant. All voltages in equation (1) are voltage drops. The voltage rise due to self-inductance is $-L\frac{di}{dt}$. The drop is, therefore, $+L\frac{di}{dt}$.

The voltage e in equation (1) is the voltage drop across the circuit. The component of this voltage to balance the ohmic reaction is ri. $L\frac{di}{dt}$ is the component to balance the reaction due to self-inductance, *i.e.*, the electromotive force of self-induction. If instead of considering reactions, the energy relations are considered, the following evidently holds:

$$eidt = ri^2dt + L\frac{di}{dt}idt \tag{2}$$

Consider the total energy concerned in a time T equal to the periodic time for the circuit in question. This is given in the following equation:

$$\int_0^T eidt = \int_0^T ri^2dt + \int_0^T L\frac{di}{dt}idt \tag{3}$$

The first term of the second member of equation (3) is the energy dissipated as heat. The second term represents the energy stored in the magnetic field. It is clear that if the self-inductance of the circuit is constant, this second term becomes zero when integrated over a complete cycle, as would be expected, since the amount of energy delivered to the field by the circuit during one-half of any complete period is balanced by the amount of energy delivered to the circuit by the field during the other half period. *It is obvious from equation* (3) *that the average power in a circuit having constant resistance and constant self-inductance is equal to the square of the effective value of the current multiplied by the resistance.*

Solution of the Differential Equation of a Circuit Containing Constant Resistance and Constant Self-inductance in Series.— The solution of the equation

$$e = ri + L\frac{di}{dt} \tag{4}$$

giving the current in terms of e, r, L and t, evidently depends for its form on e.

CASE I. GROWTH OF CURRENT IN AN INDUCTIVE CIRCUIT ON WHICH A CONSTANT ELECTROMOTIVE FORCE IS IMPRESSED.— (Fig. 32, key on a.)

$$e = E = \text{constant}$$

$$E = ri + L\frac{di}{dt} \tag{5}$$

This is a linear differential equation of the first order with constant coefficients. The general solution of such an equation is of the form

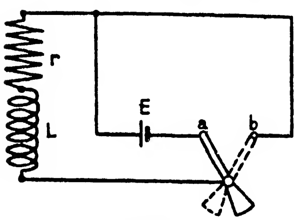


FIG. 32.

$$i = Y + u \tag{6}$$

where Y is the complementary function and u is the particular integral. (Murray, Differential Equations.) In the case of an electric circuit, Y is the transient term in the expression for the current. This term persists only during the establishment of the current and becomes zero when steady conditions have been reached. The particular integral represents the steady state, *i.e.*, the condition in the circuit after the transient term has become zero. The general equation for the establishment of a current in an electric circuit always contains a transient and a steady term. The conditions in the circuit at any instant are presented by the sum of the two terms. Usually the transient term is sensibly equal to zero after a short interval of time.

Since the impressed voltage E in equation (5) is constant, the current must follow Ohm's law after steady conditions have been established. The particular integral u, therefore, must be equal to $\frac{E}{r}$.

The complementary function is always found by making the impressed voltage zero, in this case by making E zero. The solution for the current is of the form $i = A\epsilon^{at}$. (Murray, Differential Equations.) Substituting $E = 0$ and $i = A\epsilon^{at}$ in equation (5) gives

$$0 = rA\epsilon^{at} + aAL\epsilon^{at} \tag{7}$$

This holds for all values of t.

$$0 = r + aL \tag{8}$$

Equation (8) is an equation of the first degree and therefore has but a single root.

$$a = -\frac{r}{L}$$

The complete solution of equation (5) is, therefore,

$$i = A\epsilon^{-\frac{rt}{L}} + \frac{E}{r} \tag{9}$$

where A is a constant of integration which must be determined from the conditions existing in the circuit at the instant $t = 0$. When t is zero, i in most cases is also zero. Putting both t and i equal to zero gives

$$A = -\frac{E}{r}$$

Therefore,

$$i = \frac{E}{r} - \frac{E}{r}\epsilon^{-\frac{rt}{L}} \tag{10}$$

$$= \frac{E}{r}\left(1 - \epsilon^{-\frac{rt}{L}}\right) \tag{11}$$

At the instant of closing the circuit, t is zero and the current i is also zero. The rate of change of current, $\frac{di}{dt}$, is a maximum and is equal to $\frac{E}{L}$. [Equation (5).] As t increases, $\frac{di}{dt}$ decreases, approaching zero as a limit, while the current approaches the limiting or Ohm's-law value.

If the conditions in the circuit are such that the current is not zero when t is zero, *i.e.*, at the instant the electromotive force E is applied to the circuit, the constant A may still be evaluated. Let I_0 be the current in the circuit at the instant E is added, and let E_0 be the electromotive force producing this current. Then, from equation (9), when $t = 0$,

$$I_0 = A + \frac{E_0 + E}{r}$$

$$A = I_0 - \frac{E_0 + E}{r}$$

Therefore, in this case,

$$i = \left(I_0 - \frac{E_0 + E}{r}\right)\epsilon^{-\frac{rt}{L}} + \frac{E_0 + E}{r} \qquad (12)$$

If the current I_0 has reached its steady value, *i.e.*, its Ohm's-law value $\frac{E_0}{r}$, when E is added, equation (12) becomes

$$i = -\frac{E}{r}\epsilon^{-\frac{rt}{L}} + \frac{E_0 + E}{r} \qquad (13)$$

Equation (13) reduces immediately to equation (11) when I_0 and E_0 are zero, *i.e.*, when there is no current or voltage in the circuit when E is impressed.

It cannot be emphasized too strongly that the expression for the current is always made up of two terms. One $\left[\frac{E}{r}\right.$, equation (10), and $\frac{E_0 + E}{r}$, equation $\left.(12)\right]$ gives the value of the current after steady conditions have been reached. The other $\left[\frac{E}{r}\epsilon^{-\frac{rt}{L}}\right.$, equation (10), and $\left(I_0 - \frac{E_0 + E}{r}\right)\epsilon^{-\frac{rt}{L}}$, equation $\left.(12)\right]$ is a transient which decreases logarithmically and becomes zero when steady conditions have been attained. When any change whatsoever is made in an electric circuit, whether it is in its constants or in its applied voltage, the expression for the current contains two terms, one representing the transient, the other the steady state. The coefficient of the transient term is always

determined by the conditions existing in the circuit at the instant the electromotive force is impressed and by the steady state. [See equation (12).]

Theoretically it takes an infinite time for the transient term to become zero, but in practice it usually becomes negligibly small in a comparatively brief interval of time.

The rate of increase of current depends on the ratio of the self-inductance to the resistance, *i.e.*, on the ratio of L to r. Either large resistance or small inductance makes the rate of increase of current rapid. As this rate of increase is determined by the ratio of L to r, $\frac{L}{r}$ is known as the time constant of the circuit. $\frac{L}{r}$ represents the time required for the current to reach 0.632 of its final or Ohm's-law value.

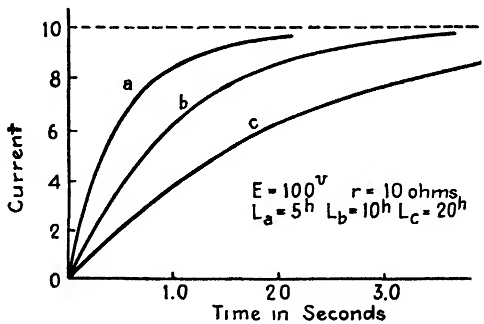


FIG. 33.

Steady and transient terms similar to those in equations (10) and (12) are typical and occur in all equations for current in circuits containing resistance and inductance, resistance and capacitance, or resistance, inductance and capacitance, in which the current has not reached its steady state. The growth of current in a circuit containing resistance and self-inductance in series is shown graphically in Fig. 33. In this figure the current is assumed to be zero when t is zero.

All three curves shown are for the same impressed voltage E and for the same resistance r. The self-inductance L is least in curve a and greatest in curve c.

Energy in the Electromagnetic Field.—*From the energy* equation [equation (3), page 129], it will be seen that energy is being constantly dissipated in heat and also is being stored in the field at a rate $L\frac{di}{dt}$. This rate is a maximum when the current reaches one-half its maximum value. The stored energy in the magnetic field at a time at which the current is I is

$$\int_0^t L\frac{di}{dt}i\,dt = \int_0^I Li\,di = \frac{1}{2}LI^2 \tag{14}$$

The expression $\frac{1}{2}LI^2$ is the electrokinetic energy of the circuit. It is analogous to the expression $\frac{1}{2}MV^2$ for the kinetic energy of a moving body, where M is the mass of the body and V is its velocity.

The expression

$$W = \frac{1}{2}LI^2 \tag{15}$$

for the electrokinetic energy of a circuit holds only when the self-inductance L is constant. If the circuit contains magnetic material, L is not constant but is a function of the current in the circuit. If $L = \mathrm{f}(i)$,

$$W = \int_0^I \mathrm{f}(i)i\,di \tag{16}$$

The flux density in a magnetic circuit containing iron may be expressed, approximately, in terms of the current by the following empirical equation,

$$\mathcal{B} = \frac{Ni}{k + k'Ni} \tag{17}$$

where the k's are constants which may be obtained readily from a magnetization curve, $\mathcal{B}$ is the flux density and N is the number of turns in the winding on the magnetic circuit. If A is the cross section of the magnetic circuit,

$$L = NA\frac{d\mathcal{B}}{di}. \tag{18}$$

Equations (17) and (18) assume that there is no leakage of flux between turns, *i.e.*, that all the flux produced by any one turn of the winding links all of the turns. This condition is never exactly fulfilled in practice, although in some cases it is approximately attained.

Case II. Decay of Current in an Inductive Circuit When the Impressed Electromotive Force Is Removed by Short Circuiting.—(Fig. 32, page 130, key on b.) The switch is assumed to be so arranged as not to break the circuit when it is thrown from a to b.

When E is zero, equation (4), page 130, becomes

$$0 = ri + L\frac{di}{dt} \tag{19}$$

The solution of this equation, as in Case I, is of the form

$$i = Y + u$$

but now the particular integral u, which represents the steady state, is zero. The complete solution of equation (19), therefore, contains only the transient term and is

$$i = A\epsilon^{-\frac{rt}{L}} \tag{20}$$

where, as before, A is the constant of integration which may be evaluated by putting t equal to zero. Assume the current to have attained a value I_0' when the electromotive force is removed. Then when $t = 0$, $i = I_0'$ and

$$A = i = I_0'$$

A is equal to the value of the current in the circuit at the instant the electromotive force is removed. Putting the value of A in equation (20) gives

$$i = I_0'\epsilon^{-\frac{rt}{L}} \tag{21}$$

If the current has reached its Ohm's-law value before the electromotive force is removed, equation (21) becomes

$$i = \frac{E}{r}\epsilon^{-\frac{rt}{L}} \tag{22}$$

From the energy equation [equation (2), page 129],

$$0 = ri^2dt + L\frac{di}{dt}idt \tag{23}$$

or

$$ri^2dt = -Lidi$$

The energy in the field is gradually being dissipated in heat in the circuit.

The conditions represented by equation (22) are shown graphically in Fig. 34.

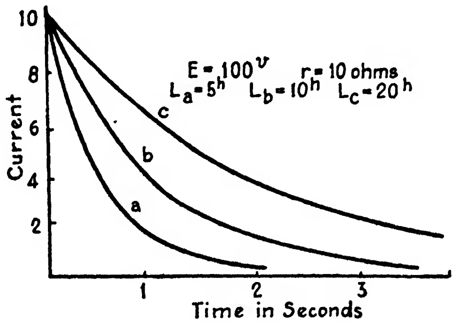


FIG. 34.

The rate of decay of the current depends on the ratio of self-inductance to resistance, *i.e.*, on the time constant $\frac{L}{r}$. Either small self-inductance or large resistance makes the decay of current rapid. The time constant represents the time required for the current in a circuit to reach 0.632 of its Ohm's-law value when the circuit is closed, or to fall to $(1 - 0.632) = 0.368$ of its Ohm's-law value when the electromotive force is removed. The above statements regarding the time constant assume that the current is zero when the circuit is closed and has reached its Ohm's-law value when the electromotive force is removed.

Breaking an Inductive Circuit.—If instead of removing the electromotive force, *i.e.*, short-circuiting it, as in Case II, the circuit is broken, the decay of current is much more rapid, owing to the great increase in resistance caused by the introduction of the air gap at the break. The energy of the magnetic field must be delivered to the circuit in the brief interval of time

required to interrupt the current. The electromotive force induced when the circuit is broken may be very great and, indeed, may be sufficiently high to establish an arc across the terminals of the circuit, even if the electromotive force impressed initially on the circuit to produce the current would not be sufficient to start the arc. If it were possible to break a circuit in zero time, without the dissipation of any energy at the break, the self-induced voltage would be infinite. In this case there would be infinite voltage across the break. Such a condition is not possible in practice, but very high voltages are produced whenever inductive circuits are broken rapidly. Special precautions must be taken when highly inductive circuits, such as the fields of motors and generators, are opened.

If an inductive circuit is shunted by a suitable non-inductive resistance before being disconnected from the mains, the rise in voltage across its terminals may be limited to any desired amount. If the shunt could be made absolutely non-inductive and its resistance were equal to the resistance of the inductive circuit to be broken, no rise in voltage could occur across the terminals of the inductive circuit even if the switch disconnecting it with its shunt from the mains could be opened in zero time. If the switch could be opened in zero time, the current in the inductive circuit would be immediately established in the shunt and would then decrease at a rate determined by equation (21), page 135, where r is the resistance of the shunt plus the resistance of the inductive circuit. The highest possible voltage across the inductive circuit before it is opened is $Ir = E$, where r is its resistance and E the voltage of the circuit to which it is connected. This voltage would be reached if the current had attained its Ohm's-law value, $I = \frac{E}{r}$, before breaking the circuit. Under this condition, the current in the shunt the instant after opening the circuit would be $I = \frac{E}{r}$. The voltage across its terminals would be Ir_s, where r_s is the resistance of the shunt. This voltage would be equal to E if r_s were made equal to r, the resistance of the inductive circuit. When the field circuits of large motors or generators have to be disconnected from the mains while carrying current, it is customary to provide them

with suitable non-inductive shunts to prevent a dangerous rise in voltage. Such shunts are called field-discharge resistances. They are used with special field-discharge switches which connect them in circuit only while the current is being interrupted.

An inductive circuit with a shunting non-inductive resistance is shown in Fig. 35. L is the inductive circuit and R is the shunting resistance. The switches a and b are interconnected in such a way that b closes before a opens.

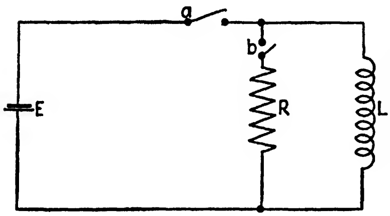


FIG. 35.

CASE III. SIMPLE HARMONIC ELECTROMOTIVE FORCE IMPRESSED ON A CIRCUIT CONTAINING CONSTANT RESISTANCE AND CONSTANT SELF-INDUCTANCE IN SERIES.—In this case, $e = E_m \sin(\omega t + \alpha)$ and equation (4), page 130, become

$$E_m \sin(\omega t + \alpha) = ri + L\frac{di}{dt} \tag{24}$$

This is a linear differential equation of the first order with constant coefficients. Its first term is a function of t. The solution of the equation is again of the form

$$i = Y + u$$

The transient term Y is found as in Cases I and II. From equation (9), the transient is

$$Y = A\epsilon^{-\frac{rt}{L}} \tag{25}$$

The constant A is determined by the conditions in the circuit at the instant $t = 0$.

The particular integral may be evaluated most easily in the following manner. Under steady conditions, the drop in voltage across the circuit is made up of two parts, one, ri, due to the resistance, the other, $L\frac{di}{dt}$, due to the self-inductance. If the resistance is constant, the drop caused by it is of the same wave form as the current. The drop caused by the self-induc-

tance L is not, however, of the same wave form as the current, even when L is constant, except when the current is sinusoidal. If the current is sinusoidal and L is constant, the drop $L\frac{di}{dt}$ is also sinusoidal, since the derivative of the current with respect to time is then a cosine function of time which is equivalent to a sine function advanced 90 degrees in phase. Since the sum of any number of sinusoidal waves of the same frequency is also a sinusoidal wave of like frequency, it follows that, if r and L are both constant and the current is sinusoidal, the voltage drop impressed on the circuit must also be sinusoidal, since it is the sum of the two sinusoidal drops, in quadrature with each other, due to the current. Conversely, if the voltage impressed on the circuit is sinusoidal, the sum of the reactions due to the current must be sinusoidal. The only way the sum of these reactions can be sinusoidal when r and L are constant is for the current to be sinusoidal.

Let the current be

$$i = I_m \sin (\omega t + \alpha - \theta) \tag{26}$$

where I_m is the maximum value of the current after steady conditions have been attained and θ is its angle of lag with respect to the voltage E_m.

Substituting the value of the current given by equation (26) in equation (24) gives

$$\begin{aligned} E_m \sin (\omega t + \alpha) &= rI_m \sin (\omega t + \alpha - \theta) \\ &\quad + L\frac{d}{dt} I_m \sin (\omega t + \alpha - \theta) \\ &= rI_m \sin (\omega t + \alpha - \theta) \\ &\quad + \omega L I_m \cos (\omega t + \alpha - \theta) \end{aligned} \tag{27}$$

Since a cosine function of time leads a sine function by 90 degrees, equation (27) may be written in the following form:

$$\begin{aligned} E_m \sin (\omega t + \alpha) &= rI_m \sin (\omega t + \alpha - \theta) \\ &\quad + \omega L I_m \sin (\omega t + \alpha - \theta + 90^\circ) \end{aligned} \tag{28}$$

The two terms in the right-hand member of equation (28) are two quadrature components of the voltage drop, $E_m \sin (\omega t + \alpha)$, across the circuit. The maximum values of these components

are rI_m and ωLI_m. The component ωLI_m leads the component rI_m by 90 degrees. These two components are, respectively, the reactive and the active components of the voltage drop.

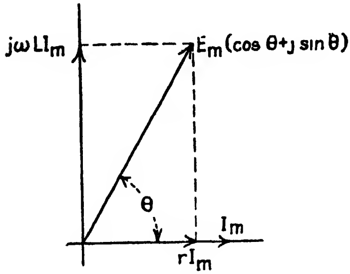


FIG. 36.

The vectors corresponding to the three terms of equation (28) are plotted in Fig. 36 for the instant of time $t = -\frac{\alpha - \theta}{\omega}$. At this instant the vector representing the current lies along the axis of reals.

The waves corresponding to the vectors are plotted in Fig. 37, θ being assumed greater than α.

Referring to Fig. 36, it is obvious that

$$E_m = \sqrt{(I_m r)^2 + (\omega LI_m)^2}$$
$$= I_m\sqrt{r^2 + \omega^2 L^2} \tag{29}$$

and

$$I_m = \frac{E_m}{\sqrt{r^2 + \omega^2 L^2}} \tag{30}$$

$$\theta = \tan^{-1}\frac{\omega L}{r} \tag{31}$$

The voltage E_m leads the current I_m by an angle θ, or the current lags the voltage by the same angle. Since θ in equations (27) and (28) and in Fig. 36 is the phase angle of the current with respect to the voltage, a minus sign is placed in front of it, as θ, as given by equation (31), is positive and is the angle of lead of voltage with respect to current.

The particular integral or steady-term in the general equation for the current in a circuit having constant resistance and constant self-inductance in series is, consequently,

$$i = \frac{E_m}{\sqrt{r^2 + \omega^2 L^2}} \sin\left(\omega t + \alpha - \tan^{-1}\frac{\omega L}{r}\right) \tag{32}$$

The complete solution of equation (24) is, therefore,

$$i = A\epsilon^{-\frac{rt}{L}} + \frac{E_m}{\sqrt{r^2 + \omega^2 L^2}} \sin\left(\omega t + \alpha - \tan^{-1}\frac{\omega L}{r}\right) \tag{33}$$

When t is zero, the current i is also zero. Putting both t and i equal to zero in equation (33) gives

$$A = \frac{-E_m}{\sqrt{r^2 + \omega^2 L^2}} \sin\left(\alpha - \tan^{-1}\frac{\omega L}{r}\right)$$

Therefore,

$$i = \frac{-E_m}{\sqrt{r^2 + \omega^2 L^2}} \sin\left(\alpha - \tan^{-1}\frac{\omega L}{r}\right)\epsilon^{-\frac{rt}{L}} + \frac{E_m}{\sqrt{r^2 + \omega^2 L^2}} \sin\left(\omega t + \alpha - \tan^{-1}\frac{\omega L}{r}\right) \quad (34)$$

The angle $\theta = \tan^{-1}\frac{\omega L}{r}$ is the angle of lag of the current behind the voltage after steady conditions have been attained. It is determined by the constants of the circuit and theoretically,

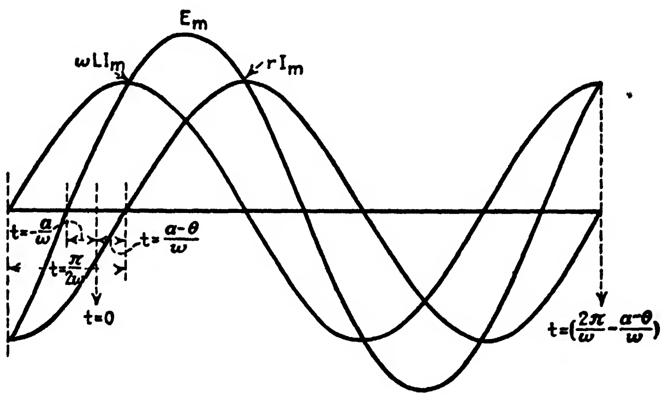


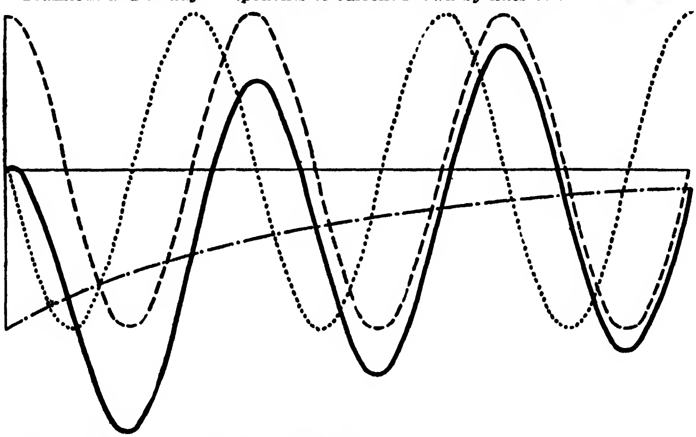
FIG. 37.

at least, may have any value between 0 and 90 degrees. It is zero when the inductance is zero, *i.e.*, for a non-inductive circuit. It would be 90 degrees for a circuit having inductance but no resistance, if such a condition were possible of attainment. In practice, θ may be very nearly 90 degrees but it can never be exactly 90 degrees, since it is impossible to have a circuit without some resistance. There is no difficulty in making a circuit practically non-inductive for ordinary frequencies.

The magnitude of the transient term in the expression for the current [first term in the second member of equation (34)] is deter-

mined by the particular point on the electromotive-force wave at which the circuit is closed. The magnitude of the phase α, in the expression for the instantaneous voltage [equation (24), page 138], fixes the value of the voltage e when t is zero. The transient is a maximum when $(\alpha - \theta) = \frac{\pi}{2}$, *i.e.*, when the circuit is closed at that point on the electromotive-force wave which

Impressed voltage shown by line of round dots.
Current shown by full line.
Transient and steady components of current shown by lines of short dashes.



$\frac{L}{r} = 0.0955$, Frequency = 25 cycles, $\frac{x}{r} = 15$, Angle of lag = 86.2 degrees.
Curves show the conditions existing when the circuit is closed at the point on the voltage wave which makes the transient a maximum, $\alpha - \theta = \frac{\pi}{2}$

FIG. 38.

corresponds to maximum current after steady conditions have been established. The transient is zero when $(\alpha - \theta) = 0$, *i.e.*, when the circuit is closed at that point on the electromotive-force wave which corresponds to zero current after steady conditions have been reached.

Figure 38 shows the resultant current and its transient and steady components when a circuit containing constant resistance and constant self-inductance in series is closed at the point on the electromotive-force wave which makes the transient a maximum. The figure is for a 25-cycle circuit having a ratio of ωL to r of 15.

It is obvious from Fig. 38 that the maximum value the current can attain, when a circuit having constant resistance and constant

self-inductance in series is closed, can never be equal to twice the maximum value of the current under steady conditions. It may reach nearly twice that value in a high-frequency circuit having a large time constant, *i.e.*, having a large ratio of self-inductance to resistance, for under such conditions the transient will have diminished but little at the end of the first half cycle and will then add to the steady component to give a maximum nearly equal to twice the maximum value of the current under steady conditions.

Under ordinary conditions, the transient is of little importance when circuits having constant resistance and constant self-inductance in series are closed, since it practically disappears after a few cycles and, in general, produces no dangerous rise in current. In the case plotted in Fig. 38, the transient has relatively little effect after two and a half or three cycles, *i.e.*, after about a tenth of a second, and this is for a circuit having a relatively large ratio of L to r.

The effect of the transient is to displace the axis of the current wave so that it lies along a logarithmic curve, which is the transient, instead of lying along the axis of time.

Although the transient is of relatively little importance in most cases, it is of importance in switching operations on high-voltage transmission lines. It is of especial importance in certain cases where the inductance is not constant as, for example, when an alternator is short-circuited or when a large transformer is switched on a line. When an alternator is short-circuited, the initial maximum value of the short-circuit current may reach ten or more times the maximum value of the sustained short-circuit current, which itself may be several times the rated full-load current. This large transient short-circuit current is caused, in the case of the alternator, by a very great decrease in the apparent inductance of the armature winding during the transient period of the short circuit.

The steady current for a circuit having constant resistance and constant self-inductance in series is given by

$$\begin{aligned} i &= \frac{E_m}{\sqrt{r^2 + \omega^2 L^2}} \sin\left(\omega t + \alpha - \tan^{-1}\frac{\omega L}{r}\right) \\ &= I_m \sin(\omega t + \alpha - \theta) \qquad (35) \end{aligned}$$

The maximum value of the current is found by dividing the maximum value of the voltage by $\sqrt{r^2 + \omega^2 L^2}$. It lags the maximum value of the voltage by an angle θ whose tangent is equal to the ratio of ωL to r. Since the effective value of a sinusoidal current is equal to its maximum value divided by the square root of two, it is evident that the effective value of the current is given by the effective value of the voltage divided by $\sqrt{r^2 + \omega^2 L^2}$. The quantity $\sqrt{r^2 + \omega^2 L^2}$ plays a part in alternating-current circuits similar to that of resistance in direct-current circuits. If it is denoted by the letter z, the expression for current may be written in a form similar to Ohm's law for direct currents, *i.e.*,

$$I = \frac{E}{\sqrt{r^2 + \omega^2 L^2}} = \frac{E}{z} \tag{36}$$

Impedance and Reactance of a Series Circuit Containing Constant r and Constant L.—The quantity

$$z = \sqrt{r^2 + \omega^2 L^2}$$

is called the *impedance* and is measured in ohms. It is a constant only when the resistance, self-inductance and frequency are constant. The expression ωL is called the *inductive reactance* and is also measured in ohms. Reactance is denoted by the letter x. Thus $\omega L = x$ and

$$z = \sqrt{r^2 + x^2} \tag{37}$$

Inductive reactance is constant only when frequency and self-inductance are constant. It is the inductive reactance of a circuit-which causes the current to lag in phase behind the voltage and thus to have a quadrature component. It is because this component is caused by reactance that it is called the *reactive* component.

Vector Method of Determining the Steady Current for a Circuit Having Constant Resistance and Constant Self-inductance in Series.—The current in amperes will be taken along the axis of reals. Refer to Fig. 36, page 140. For the present purpose, read I_m and E_m on the figure as I and E, respectively; *i.e.*, as root-mean-square or effective values instead of maximum values. The active and reactive components of the impressed

electromotive force are then given by rI and $j\omega LI = jxI$, respectively.

$$\bar{E} = r\bar{I} + jx\bar{I} = \bar{I}(r + jx) = \bar{I}\bar{z}$$
$$\bar{I} = \frac{\bar{E}}{r + jx}$$

In complex, therefore, the impedance of an inductive circuit is given by

$$\bar{z} = r + jx \tag{38}$$

The magnitude of z in ohms is $\sqrt{r^2 + x^2}$.

It should be noted that impedance is not a vector but a complex quantity. When it is multiplied by vector current, vector voltage results. Complex quantities have no reference axis in the ordinary sense, but, when they are multiplied or divided by a vector such as $\bar{A}$, the result is a new vector referred to the same axis as the vector $\bar{A}$, usually not in phase with $\bar{A}$. Although complex quantities, such as impedance, are not vectors, when multiplied or divided, added or subtracted, they must be treated as vectors in so far as these operations are concerned.

Polar Expression for the Impedance of a Circuit Containing Constant Resistance and Constant Self-inductance in Series.— Multiplying and dividing the complex expression for impedance by $\sqrt{r^2 + x^2}$ does not alter its value.

$$\begin{aligned}\bar{z} &= (r + jx)\,\frac{\sqrt{r^2 + x^2}}{\sqrt{r^2 + x^2}} \\ &= \sqrt{r^2 + x^2}\left(\frac{r}{\sqrt{r^2 + x^2}} + j\frac{x}{\sqrt{r^2 + x^2}}\right) \\ &= z\,(\cos\theta + j\sin\theta)\end{aligned} \tag{39}$$

where z is the magnitude of the impedance and θ is its angle, *i.e.*, the angle which is determined by the relation $\theta = \tan^{-1}\dfrac{x}{r}$.

From equation (39) it is seen that impedance is a scalar quantity multiplied by the operator $(\cos\theta + j\sin\theta)$ which rotates through the angle θ.

Multiplying current expressed in complex by impedance, also expressed in complex, therefore gives the correct value of the

impedance drop rotated into the correct phase position with respect to the current.

From equation (39) it is obvious that the polar expression for impedance is

$$\bar{z} = z\underline{|\theta} \tag{40}$$

where $\theta = \tan^{-1}\frac{x}{r}$. For inductive impedance, θ is positive.

Circuit Containing Constant Resistance and Constant Capacitance in Series.—The condition of a circuit containing constant resistance and constant capacitance in series is completely determined by the equation

$$e = ri + \frac{q}{C} \tag{41}$$

where e is the impressed voltage drop. The component of this to supply the ohmic drop is ri. $\frac{q}{C}$ is the component to overcome the counter electromotive force of the condenser. If q is the charge on the condenser in coulombs and C is the capacitance of the condenser in farads, $\frac{q}{C}$ is the voltage drop across the condenser in volts.

The energy relation is shown by

$$eidt = ri^2dt + \frac{q}{C}idt \tag{42}$$

Consider the total energy concerned in any given time T,

$$\int_0^T eidt = \int_0^T ri^2dt + \int_0^T \frac{q}{C}idt \tag{43}$$

The first term of the second member is the energy dissipated in joule heating, *i.e.*, in the i^2r loss. The second term represents the energy stored in the electrostatic field as potential energy due to the strain in the dielectric of the condenser.

Solution of the Differential Equation for a Circuit Containing Constant Resistance and Constant Capacitance in Series.—The solution of equation (41), giving the current and charge of the condenser in terms of e, r, C and t, evidently depends for its form on the impressed electromotive force.

CASE I. GROWTH OF CHARGE AND CURRENT IN A CAPACITIVE CIRCUIT ON WHICH A CONSTANT ELECTROMOTIVE FORCE IS IMPRESSED.—(Figure 39, key on a.)

$$e = E = \text{constant}$$

$$E = ri + \frac{q}{C}$$

Since $i = \dfrac{dq}{dt}$,

$$E = r\frac{dq}{dt} + \frac{q}{C} \qquad (44)$$

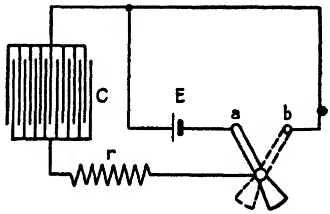


FIG. 39.

This is a linear differential equation of the first order. It is of the same form as equation (5), page 130, and its solution is of the same type.

$$q = Y + u$$

Obviously, the steady state is represented by the final charge on the condenser. The particular integral u is therefore equal to EC. The transient Y is found as before by putting $E = 0$. The solution of the equation is of the form $q = A\epsilon^{at}$.

Substituting $E = 0$ and $q = A\epsilon^{at}$ in equation (44) gives

$$0 = raA\epsilon^{at} + A\frac{\epsilon^{at}}{C} \qquad (45)$$

This holds for all values of t.

$$0 = ra + \frac{1}{C} \qquad (46)$$

Equation (46) is an equation of the first degree and therefore has but a single root.

$$a = -\frac{1}{Cr}$$

The complete solution of equation (44) is, therefore,

$$q = A\epsilon^{-\frac{t}{Cr}} + EC \qquad (47)$$

The constant of integration is found by putting t equal to zero. Let Q_0 be the charge on the condenser when the circuit is closed, *i.e.*, when t is zero. Then

$$A = Q_0 - EC \qquad (48)$$

Putting this value of the constant of integration, A, in equation (47) gives

$$q = EC + (Q_0 - EC)\epsilon^{-\frac{t}{Cr}} \tag{49}$$

If the initial charge is zero, equation (49) becomes

$$\begin{aligned} q &= EC\left(1 - \epsilon^{-\frac{t}{Cr}}\right) \\ &= Q\left(1 - \epsilon^{-\frac{t}{Cr}}\right) \end{aligned} \tag{50}$$

where $Q = EC$ is the final charge on the condenser.

When the initial charge on the condenser is zero, the counter electromotive force of the condenser is zero when $t = 0$. The current in the circuit at this instant has its maximum value, $\frac{E}{r}$. As t increases, q also increases, producing a counter electromotive force $\frac{q}{C}$, which cuts down the current and hence the rate at which the charge increases, and therefore diminishes the rate of decrease of the current. The charge approaches the limit $Q = EC$, while the current approaches the limit zero.

The rate of increase of charge and, therefore, the rate of decrease in current is determined by Cr, the time constant of the circuit. Like the time constant for a circuit containing constant resistance and constant self-inductance in series, it is equal to the time required for the charge to reach 0.632 of its final or maximum value.

Since $i = \frac{dq}{dt}$, the equation for current may be found by differentiating equation (49) with respect to time.

$$i = -\left(\frac{Q_0 - EC}{Cr}\right)\epsilon^{-\frac{t}{Cr}} \tag{51}$$

$$= -\frac{Q_0}{Cr}\epsilon^{-\frac{t}{Cr}} + \frac{E}{r}\epsilon^{-\frac{t}{Cr}} \tag{52}$$

When the initial charge on the condenser is zero, equations (51) and (52) reduce to

$$i = \frac{E}{r}\epsilon^{-\frac{t}{Cr}} \tag{53}$$

When the initial charge on the condenser is zero, the charge on the condenser increases logarithmically to the limiting value EC [equation (50)]. The current, on the other hand, reaches its maximum value $\frac{E}{r}$ instantly and then diminishes logarithmically to zero.

The growth of the charge and the decay of the current, when the initial charge is zero, are shown in Fig. 40. The curves are

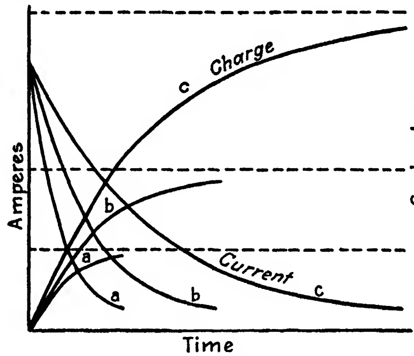


FIG. 40.

all for the same impressed voltage E and the same resistance r. The capacitance is least for curve a and greatest for curve c.

Energy of the Electrostatic Field.—From the energy equation [equation (42), page 146], it may be seen that energy is constantly being dissipated in heat and also is being stored in the electrostatic field at a rate $\frac{q}{C}i$, which constantly decreases. The total energy thus stored in the electrostatic field is evidently given by

$$\int_0^\infty \frac{q}{C} i\,dt = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CE^2 \tag{54}$$

where Q and E are, respectively, the final charge and the final voltage of the condenser. The expression $\frac{1}{2}CE^2$ represents the electropotential energy of the condenser due to its charge. It corresponds to the potential energy in mechanics of a stretched spring or other elastic body which is under stress.

When a condenser is charged through a fixed resistance from a constant potential, one-half of the energy given to the circuit

is absorbed as i^2rt loss or joule heating in the resistance. The other half is stored as electropotential energy in the condenser. The efficiency of a system involving the charging of a condenser from a constant-potential source through a fixed resistance, therefore, cannot be greater than fifty per cent.

Putting T in the energy equation [equation (43), page 146] equal to infinity gives

$$E\int_0^\infty i\,dt = \int_0^\infty ri^2dt + \frac{1}{C}\int_0^\infty qi\,dt$$

$$E\int_0^\infty \frac{E}{r}\epsilon^{-\frac{t}{Cr}}dt = r\int_0^\infty \left(\frac{E}{r}\right)^2 \epsilon^{-\frac{2t}{Cr}}dt + \frac{1}{C}\int_0^Q qdq$$

$$E\left\{\frac{E}{r}(-Cr)\,\epsilon^{-\frac{t}{Cr}}\right\}_0^\infty = r\left\{\frac{E^2}{r^2}\left(-\frac{Cr}{2}\right)\epsilon^{-\frac{2t}{Cr}}\right\}_0^\infty + \left\{\frac{q^2}{2C}\right\}_0^Q$$

$$E^2C = \frac{E^2C}{2} + \frac{E^2C}{2} \tag{55}$$

The first term of equation (55) is the energy supplied to the circuit. The second and third terms are, respectively, the energy loss in the resistance and the energy stored in the condenser. The second and third terms are each equal to one-half the total energy supplied to the circuit.

Case II. Decay of the Charge in a Capacitive Circuit When the Impressed Electromotive Force Is Removed by Short-circuiting.—(Figure 39, page 147, key on b.) Since electromotive force e is equal to zero,

$$0 = ri + \frac{q}{C} = r\frac{dq}{dt} + \frac{q}{C} \tag{56}$$

In this case, the steady charge is obviously zero since the electromotive force is zero. The term which represents the steady state in the equation for the charge is therefore zero.

$$\begin{aligned} q &= Y + u \\ &= Y + 0 \end{aligned} \tag{57}$$

The complete solution of equation (56) consists of a transient term only. The solution is

$$q = A\epsilon^{-\frac{t}{Cr}}$$

When t is equal to zero, q is equal to Q_0', the initial charge on the condenser, *i.e.*, the charge at the instant it begins to discharge. Therefore,

$$A = Q_0'$$

and

$$q = Q_0' \epsilon^{-\frac{t}{Cr}} \tag{58}$$

If the charge has reached its final value EC, where E is the charging potential, before the electromotive force is removed, *i.e.*, before the discharge is started, equation (58) becomes

$$q = EC\epsilon^{-\frac{t}{Cr}} \tag{59}$$

In this case, when $t = 0$, *i.e.*, at the instant of short-circuiting or removing the electromotive force, the charge q is equal to $Q_0' = EC$. The charge decreases logarithmically to zero.

Since $i = \frac{dq}{dt}$, the equation for current may be obtained by differentiating the equation for charge, *i.e.*, equation (58), with respect to time.

$$i = -\frac{Q_0'}{Cr}\epsilon^{-\frac{t}{Cr}} \tag{60}$$

When $Q_0' = EC$, *i.e.*, when the steady state has been reached before the condenser begins to discharge, the expression for the current becomes

$$i = -\frac{E}{r}\epsilon^{-\frac{t}{Cr}}$$

$$= -I_0'\epsilon^{-\frac{t}{Cr}} \tag{61}$$

In this case, the current i is equal to its Ohm's-law value, $I_0' = \frac{E}{r}$, when t is zero.

At the instant of removing the electromotive force, *i.e.*, when $t =$ zero, the charge on the condenser is a maximum and is equal to $Q = EC$, where E is the voltage of the condenser at the instant discharge is started. As t increases, the charge diminishes logarithmically to zero. The current has its maximum value,

$\frac{E}{r}$, for $t = 0$ and also diminishes logarithmically, approaching zero as a limit with the charge. Observe, however, that the current is negative, which means that the current is flowing out of the condenser. During the discharge the energy of the electrostatic field is being gradually dissipated in heat in the resistance of the circuit.

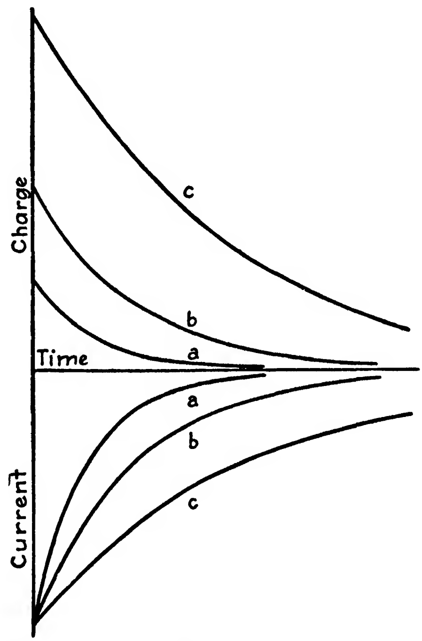


Fig. 41.

The decay of charge and current is shown graphically in Fig. 41. The capacitance is least for curve a and greatest for curve c.

Case III. Simple Harmonic Electromotive Force Impressed on a Circuit Containing Constant Resistance and Constant Capacitance in Series.—In this case, the electromotive force is $e = E_m \sin(\omega t + \alpha)$ and equation (41), page 146, becomes

$$E_m \sin(\omega t + \alpha) = r\frac{dq}{dt} + \frac{q}{C} \tag{62}$$

This is a linear differential equation of the first order with constant coefficients. The first term is a function of t. The solution is of the form

$$q = Y + u$$

The complementary function, Y, is found in the same manner as is Case I, by putting the left-hand member of the equation equal to zero. This gives the transient component of the charge. From equation (47), page 147, this is

$$Y = A\,\epsilon^{-\frac{t}{Cr}} \tag{63}$$

The particular integral, *i.e.*, the term representing the steady state, may be found by a method similar to that used in Case III, page 138, for an inductive circuit with a sinusoidal electromotive force impressed.

Under steady conditions, the voltage drop across the circuit is made up of two parts, one, $ri = r\frac{dq}{dt}$, to supply the resistance drop, the other, $\frac{q}{C}$, caused by the charge on the condenser.

When the charge varies sinusoidally with time, the resistance drop is also sinusoidal in form, provided the resistance is constant, for then the drop is equal to a constant multiplied by the derivative of a sine function. The derivative of a sine function is a cosine function, which is equivalent to a sine function advanced 90 degrees in phase. If C is constant, the voltage drop $\frac{q}{C}$ across the condenser terminals is, obviously, of the same wave form as the charge. Since the sum of any number of sinusoidal waves of like frequency is a sinusoidal wave of the same frequency, it follows that, if r and C are both constant and the charge varies sinusoidally with time, the voltage across the circuit required to produce the charge must also be sinusoidal in wave form, since it is the sum of two sinusoidal drops related to the charge. Conversely, if the voltage impressed on the circuit is sinusoidal, the sum of the reactions due to the charge must be sinusoidal. The only way the sum of these reactions can be sinusoidal, when both r and C are constant, is for the charge to be sinusoidal.

Let the charge be

$$q = Q_m \sin(\omega t + \alpha + \theta') \qquad (64)$$

where Q_m is the maximum value of the charge after steady conditions have been attained and θ' is the phase angle of the charge with respect to the impressed electromotive force E_m.

Substituting the value of the charge given by equation (64) in equation (62) gives

$$\begin{aligned} E_m \sin(\omega t + \alpha) &= r\frac{d}{dt}\{Q_m \sin(\omega t + \alpha + \theta')\} \\ &\quad + \frac{1}{C} Q_m \sin(\omega t + \alpha + \theta') \\ &= r\omega Q_m \cos(\omega t + \alpha + \theta') \\ &\quad + \frac{1}{C} Q_m \sin(\omega t + \alpha + \theta') \qquad (65) \end{aligned}$$

Since a cosine function leads a sine function by 90 degrees, equation (65) may be written

$$\begin{aligned} E_m \sin(\omega t + \alpha) &= r\omega Q_m \sin(\omega t + \alpha + \theta' + 90°) \\ &\quad + \frac{1}{C} Q_m \sin(\omega t + \alpha + \theta') \qquad (66) \end{aligned}$$

The two components of the second member of equation (66) are quadrature components of the voltage drop $E_m \sin(\omega t + \alpha)$ across the circuit. The maximum values of these components are $r\omega Q_m$ and $\frac{Q_m}{C}$.

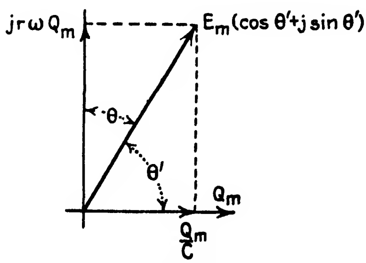


Fig. 42.

The vectors corresponding to the three terms of equation (66) are plotted in Fig. 42 for the instant of time $t = -\frac{\alpha + \theta'}{\omega}$. At this instant the vector representing the charge lies along the axis of reals.

Referring to Fig. 42, it is obvious that

$$E_m = \sqrt{\left(r\omega Q_m\right)^2 + \left(\frac{Q_m}{C}\right)^2}$$

$$= \omega Q_m \sqrt{r^2 + \frac{1}{\omega^2 C^2}} \tag{67}$$

$$Q_m = \frac{E_m}{\omega \sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \tag{68}$$

$$\theta' = \tan^{-1} \frac{r\omega}{\frac{1}{C}} = \tan^{-1} r\omega C \tag{69}$$

The voltage E_m leads the charge Q_m by an angle θ' or the charge lags the voltage by the same angle. Since θ' in equation (66) is the phase angle of the charge with respect to the voltage, it should be considered negative, as the charge lags the voltage.

The particular integral or steady term for the charge in a circuit containing constant resistance and constant capacitance in series is, consequently,

$$q = \frac{E_m}{\omega \sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \sin\,(\omega t + \alpha - \tan^{-1} r\omega C) \tag{70}$$

The complete solution of equation (62) is, therefore,

$$q = A\epsilon^{-\frac{t}{Cr}} + \frac{E_m}{\omega \sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \sin\,(\omega t + \alpha - \tan^{-1} r\omega C) \tag{71}$$

If the initial charge is zero, *i.e.*, if the condenser is uncharged when the circuit is closed, q is zero when t is zero. Putting both t and q zero in equation (71) gives

$$A = \frac{-E_m}{\omega \sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \sin\,(\alpha - \tan^{-1} r\omega C) \tag{72}$$

Therefore,

$$q = \frac{-E_m}{\omega \sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \sin\,(\alpha - \tan^{-1} r\omega C)\epsilon^{-\frac{t}{Cr}}$$

$$+ \frac{E_m}{\omega \sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \sin\,(\omega t + \alpha - \tan^{-1} r\omega C) \tag{73}$$

Since $i = \frac{dq}{dt}$, the equation for the current, when a sinusoidal electromotive force is impressed on a circuit containing constant resistance and constant capacitance in series, may be found by differentiating equation (73) with respect to t.

$$i = \frac{E_m}{r\omega C\sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \sin(\alpha - \tan^{-1} r\omega C)\epsilon^{-\frac{t}{Cr}}$$
$$+ \frac{E_m}{\sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \cos(\omega t + \alpha - \tan^{-1} r\omega C) \quad (74)^*$$

Replacing θ' by $90° - \theta$ (see Fig. 42) and also remembering that $\cos(\beta - 90°) = \sin\beta$ and $\sin(\beta - 90°) = -\cos\beta$, gives

$$i = \frac{-E_m}{r\omega C\sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \cos\left(\alpha + \tan^{-1}\frac{1}{r\omega C}\right)\epsilon^{-\frac{t}{Cr}}$$
$$+ \frac{E_m}{\sqrt{r^2 + \frac{1}{\omega^2 C^2}}} \sin\left(\omega t + \alpha + \tan^{-1}\frac{1}{r\omega C}\right) \quad (75)$$

where $\tan^{-1}\frac{1}{r\omega C} = \theta$ is the angle of lead of the current with respect to the impressed voltage after steady conditions have been established.

It should be observed that θ is positive, and, therefore, after steady conditions have been reached, the current leads the impressed electromotive force in a circuit containing constant resistance and constant capacitance in series by an angle whose tangent depends upon the resistance, capacitance and frequency of the circuit. For a circuit containing constant resistance and constant self-inductance in series, the current lags the impressed electromotive force by an angle whose tangent depends upon the resistance, self-inductance and frequency of the circuit. (See page 141.)

The limiting value a current can attain when a sinusoidal

* This equation holds only when the condenser is uncharged when $t = 0$.

electromotive force is impressed on a circuit containing constant resistance and constant self-inductance in series is twice its steady-state maximum value. (See page 142.) No such limitation exists when a sinusoidal electromotive force is impressed on a circuit containing constant resistance and constant capacitance in series. For fixed r and C, the transient is a maximum when the circuit is closed at a point on the electromotive-force wave which makes $\left(\alpha + \tan^{-1}\frac{1}{r\omega C}\right) = 0$. [See equation (75).] Under this condition, the steady term is zero when t is zero, *i.e.*, at the instant the circuit is closed. The transient has its greatest value when $t = 0$, *i.e.*, at the instant the circuit is closed, and then decreases logarithmically to zero at a rate which depends upon the time constant Cr.

The ratio of the maximum value of the transient term to the maximum value of the steady term in the current equation [equation (75)] increases as the ratio of $\frac{1}{\omega C} = x$ to r increases, and approaches infinity as a limit. The maximum value of the transient term approaches the Ohm's-law value of the current, *i.e.*, $\frac{E_m}{r}$, as the ratio of $\frac{1}{\omega C}$ to r increases, since the coefficient of the transient term becomes approximately equal to $\frac{E_m}{r}$ when r is small compared with $\frac{1}{\omega C}$.

The ratio of r to $\frac{1}{\omega C}$ at 60 cycles for a well-made commercial static condenser may be as low as 0.01 or 0.02. If a condenser having a ratio of r to $\frac{1}{\omega C}$ equal to 0.01 were connected to a constant-potential, 60-cycle circuit at the point on the electromotive-force wave which would make the transient a maximum, the current would immediately rise to approximately 100 times the value it would have after steady conditions were established.

For circuits having constants likely to be met in practice, the transient, although it may be initially very large, practically disappears in a very small fraction of a cycle. For example, in the case just mentioned, ω would be 377. Let r be 0.1 ohm.

Then,

$$\frac{r}{\dfrac{1}{\omega C}} = 0.01$$

$$C = \frac{0.01}{r \times \omega} = \frac{0.01}{0.1 \times 377} = 2.65 \times 10^{-4} \text{ farad}$$

For the transient to fall to 0.1 its initial value,

$$\epsilon^{-\frac{t}{Cr}} = 0.1$$

$$t = \frac{1}{16{,}400} \text{ second}$$

The duration of the transient is too short to be of importance.

The maximum value of the steady component of the current is found by dividing the maximum value of the voltage by $\sqrt{r^2 + \dfrac{1}{\omega^2 C^2}}$. It leads the maximum value of the voltage by an angle whose tangent is equal to the ratio of $\dfrac{1}{\omega C}$ to r, *i.e.*, by an angle whose tangent is $\dfrac{1}{r\omega C}$. Since the effective value of a sinusoidal current is equal to its maximum value divided by the square root of two, it is evident that the effective value of the current is given by the effective value of the voltage divided by $\sqrt{r^2 + \dfrac{1}{\omega^2 C^2}}$. The quantity $\sqrt{r^2 + \dfrac{1}{\omega^2 C^2}}$ plays the same part in a capacitive circuit that the quantity $\sqrt{r^2 + \omega^2 L^2}$ does in an inductive circuit.

$$I = \frac{E}{\sqrt{r^2 + \dfrac{1}{\omega^2 C^2}}} = \frac{E}{z} \tag{76}$$

Impedance and Reactance of a Circuit Containing Constant *r* and Constant *C*.—The quantity

$$z = \sqrt{r^2 + \frac{1}{\omega^2 C^2}}$$

is called the *impedance* and is measured in ohms. It is a constant only when resistance, capacitance and frequency are constant.

The expression $-\frac{1}{\omega C}$ is called the *capacitive reactance* and is also measured in ohms. Reactance is denoted by the letter x. Capacitive reactance is constant only when frequency and capacitance are constant. The capacitance of a circuit under ordinary conditions is constant.

It should be noted that inductive reactance

$$x_L = \omega L = 2\pi f L$$

is directly proportional to frequency. Capacitive reactance

$$x_C = -\frac{1}{\omega C} = -\frac{1}{2\pi f C}$$

on the other hand, is inversely proportional to frequency. Inductive reactance is positive. Capacitive reactance is negative. The significance of the negative sign with capacitive reactance will be understood from the *vector method* which follows.

Vector Method of Determining the Steady Component of the Current in a Circuit Having Constant Resistance and Constant Capacitance in Series.—The current vector, I amperes effective, is taken along the axis of reals. The effective values of the active and reactive components of the impressed electromotive force are given by rI and $-j\frac{1}{\omega C}I = jxI$, respectively. The reactive component must be negative, since the current in a capacitive circuit leads the impressed electromotive force.

$$\bar{E} = r\bar{I} + j\frac{-1}{\omega C}\bar{I} = r\bar{I} + jx\bar{I} = \bar{I}(r + jx) \qquad (77)$$

$$\bar{I} = \frac{\bar{E}}{r + j\frac{-1}{\omega C}} = \frac{\bar{E}}{r + jx}$$

It must be remembered when substituting the value of x in the above equations that capacitive reactance is defined by $\frac{-1}{\omega C}$ and that it is a negative quantity.

The vectors corresponding to equation (77) are plotted in Fig. 43.

The waves corresponding to the vectors in Fig. 43 are shown in Fig. 44. Time is taken zero when the current is zero.

In complex, therefore, the impedance of a capacitive circuit is given by

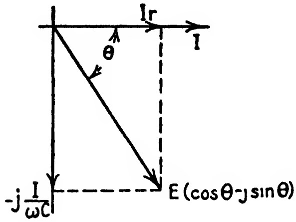


FIG. 43.

$$\bar{z} = r + j\frac{-1}{\omega C} = r + jx \qquad (78)$$

The magnitude of z in ohms is, of course, $\sqrt{r^2 + x^2}$.

Polar Expression for the Impedance of a Circuit Containing Constant Resistance and Constant Capacitance in Series.—The polar expression for the impedance of a circuit containing constant resistance and constant capacitance in series is (see page 145)

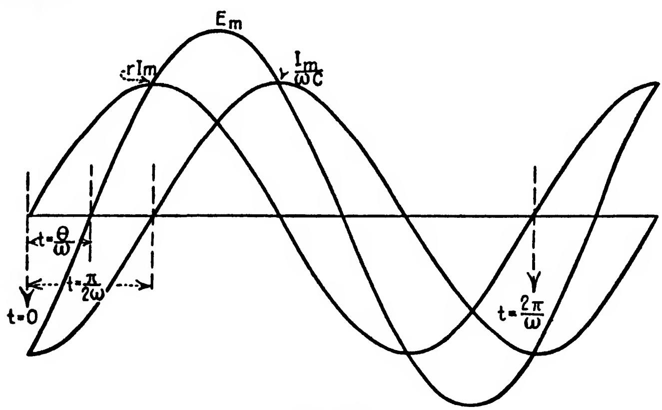


FIG. 44.

$$\bar{z} = z\underline{|\theta} \qquad (79)$$

where the angle θ is determined by the relation

$$\theta = \tan^{-1}\frac{x}{r}$$

For capacitance, x is negative. (See page 159.) Therefore θ, in the polar expression for the impedance of a circuit containing resistance and capacitance in series, is negative.

Circuit Containing Constant Resistance, Constant Self-inductance and Constant Capacitance in Series.—The conditions in a circuit containing constant resistance, constant self-inductance and constant capacitance in series are completely determined by the equation

$$e = ri + L\frac{di}{dt} + \frac{q}{C} \tag{80}$$

where e is the impressed electromotive-force drop and ri, $L\frac{di}{dt}$ and $\frac{q}{C}$ are, respectively, the components of the impressed electromotive-force drop to supply the ohmic drop, the drop due to self-induction and the drop due to the condenser. If the electromotive force e is in volts, r must be in ohms, L in henrys and C in farads. The current and charge are then in amperes and coulombs, respectively.

The energy relation corresponding to equation (80) is

$$\int_0^T eidt = \int_0^T ri^2dt + \int_0^T L\frac{di}{dt}idt + \int_0^T \frac{q}{C}idt \tag{81}$$

The first term of the second member of equation (81) is the energy dissipated in heat in the resistance of the circuit. The second term is the energy stored in the magnetic field of the inductance and the third term is the energy stored in the electrostatic field of the condenser.

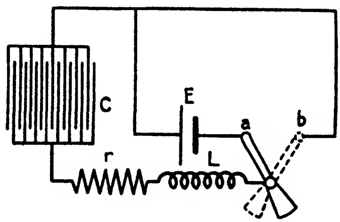


FIG. 45.

Solution of the Differential Equation for a Circuit Containing Constant Resistance, Constant Self-inductance and Constant Capacitance in Series.—The solution of equation (80) for current in terms of e, r, L, C and t evidently depends for its form upon e.

CASE I. CHARGE OF A CONDENSER THROUGH A CIRCUIT CONTAINING CONSTANT RESISTANCE AND CONSTANT SELF-INDUCTANCE IN SERIES ON WHICH IS IMPRESSED A CONSTANT ELECTROMOTIVE FORCE.—(Figure 45, key on *a*.)

$$E = ri + L\frac{di}{dt} + \frac{q}{C} \tag{82}$$

$$E = r\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} \tag{83}$$

Equation (83) is a linear differential equation of the second order with constant coefficients. Its complete solution for charge q is of the form

$$q = Y + u \tag{84}$$

where, as in all previous cases, Y and u are, respectively, the complementary function and particular integral and represent the transient and steady states. As in Case I, for a circuit having constant resistance and constant capacitance in series, the current is zero when the steady state is reached. Under this condition, the charge on the condenser is obviously constant and equal to CE. The term u in equation (84), representing the steady state, is therefore CE. The term Y, representing the complementary function or transient, is found by putting $E = 0$ and $q = A\epsilon^{at}$. (See Murray, Differential Equations.)

Making these substitutions gives

$$0 = \left(ra + La^2 + \frac{1}{C}\right)\epsilon^{at} \tag{85}$$

This holds for all values of t.

$$0 = ra + La^2 + \frac{1}{C} \tag{86}$$

Equation (86) is of the second degree and therefore has two roots. These are

$$a_1 = \frac{-rC + \sqrt{r^2C^2 - 4LC}}{2LC} \tag{87}$$

and

$$a_2 = \frac{-rC - \sqrt{r^2C^2 - 4LC}}{2LC} \tag{88}$$

The complete solution of equation (83) is, therefore,

$$q = A_1\epsilon^{a_1t} + A_2\epsilon^{a_2t} + EC \tag{89}$$

where A_1 and A_2 are arbitrary constants of integration.

Since $i = \dfrac{dq}{dt}$,

$$i = A_1 a_1 \epsilon^{a_1 t} + A_2 a_2 \epsilon^{a_2 t} \tag{90}$$

The form taken by the solutions of equations (89) and (90) depends on the relation of r^2C^2 to $4LC$. If r^2C is greater than $4L$, the roots a_1 and a_2 are both *real*. If r^2C is less than $4L$, both roots are *imaginary*. If r^2C is equal to $4L$, the roots are *equal*.

Case Ia. $r^2C > 4L$ or $r > 2\sqrt{\dfrac{L}{C}}$. In this case, it is evident that the two roots, a_1 and a_2, are essentially negative. If $t = 0$, $q = 0$ and $i = 0$. Putting these values in equations (89) and (90), and solving for the constants A_1 and A_2, gives

$$A_1 = EC\frac{a_2}{a_1 - a_2} = EC\left(\frac{-rC - \sqrt{r^2C^2 - 4LC}}{2\sqrt{r^2C^2 - 4LC}}\right) \tag{91}$$

$$A_2 = -EC\frac{a_1}{a_1 - a_2} = -EC\left(\frac{-rC + \sqrt{r^2C^2 - 4LC}}{2\sqrt{r^2C^2 - 4LC}}\right) \tag{92}$$

Hence, from equations (89) and (90),

$$q = EC - EC\left\{\frac{rC + \sqrt{r^2C^2 - 4LC}}{2\sqrt{r^2C^2 - 4LC}}\epsilon^{-\left(\frac{rC - \sqrt{r^2C^2 - 4LC}}{2LC}\right)t} - \frac{rC - \sqrt{r^2C^2 - 4LC}}{2\sqrt{r^2C^2 - 4LC}}\epsilon^{-\left(\frac{rC + \sqrt{r^2C^2 - 4LC}}{2LC}\right)t}\right\} \tag{93}$$

$$i = \frac{EC}{\sqrt{r^2C^2 - 4LC}}\left\{\epsilon^{-\left(\frac{rC - \sqrt{r^2C^2 - 4LC}}{2LC}\right)t} - \epsilon^{-\left(\frac{rC + \sqrt{r^2C^2 - 4LC}}{2LC}\right)t}\right\} \tag{94}$$

The charge on the condenser, starting at the value zero, approaches the value $Q = EC$ as its limit. The current, on the other hand, starts at the value zero, rises to a maximum and then decreases, approaching zero as its limit.

Since $i = \dfrac{dq}{dt}$, it is evident that the slope of the curve representing the charge is proportional to the current at any instant.

This curve must, therefore, have a point of inflection at the time when the current has its maximum value. In Fig. 46 are given the curves of charge and current for the following values of the constants:

$E = 2000$ volts	$L = 0.0016$ henry
$r = 100$ ohms	$C = 1$ microfarad

Case Ib. $r^2C < 4L$ or $r < 2\sqrt{\dfrac{L}{C}}$. In this case, it is evident that the roots a_1 and a_2, equations (87) and (88), page 162, are imaginary, since the radical in the expressions for both a_1 and a_2 is negative and may be written $\sqrt{-1} \times \sqrt{4LC - r^2C^2}$.

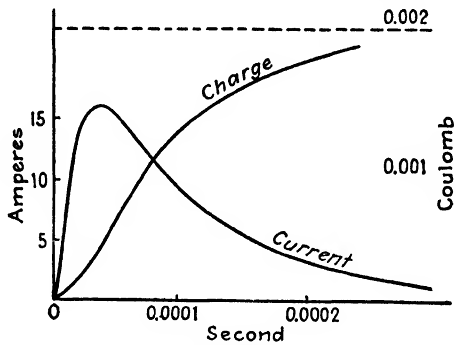


Fig. 46.

Using the operator j to indicate the imaginary $\sqrt{-1}$, the equations for a_1 and a_2 become

$$a_1 = -\frac{r}{2L} + j\frac{\sqrt{4LC - r^2C^2}}{2LC} = a + jb \tag{95}$$

and

$$a_2 = -\frac{r}{2L} - j\frac{\sqrt{4LC - r^2C^2}}{2LC} = a - jb \tag{96}$$

Here $a = -\dfrac{r}{2L}$ and $b = \dfrac{\sqrt{4LC - r^2C^2}}{2LC}$. Both a and b are real. Equation (89), page 162, now becomes

$$q = A_1\epsilon^{(a+jb)t} + A_2\epsilon^{(a-jb)t} + EC \tag{97}$$

$$= (A_1\epsilon^{jbt} + A_2\epsilon^{-jbt})\epsilon^{at} + EC \tag{98}$$

From the relations $(\cos\theta + j\sin\theta) = \epsilon^{j\theta}$ and $(\cos\theta - j\sin\theta) = \epsilon^{-j\theta}$ (see page 17), equation (98) may now be written

$$\begin{aligned} q &= \epsilon^{at}\{A_1(\cos bt + j\sin bt) + A_2(\cos bt - j\sin bt)\} + EC \\ &= (A_1 + A_2)\epsilon^{at}\cos bt + j(A_1 - A_2)\epsilon^{at}\sin bt + EC \\ &= A\epsilon^{at}\cos bt + B\epsilon^{at}\sin bt + EC \qquad (99) \end{aligned}$$

in which $A = (A_1 + A_2)$ and $B = j(A_1 - A_2)$.

From equation (99), since $i = \dfrac{dq}{dt}$,

$$i = (Aa + Bb)\epsilon^{at}\cos bt + (Ba - Ab)\epsilon^{at}\sin bt \qquad (100)$$

if

$$t = 0,\ q = 0 \text{ and } i = 0$$

Therefore

$$A = -EC$$

$$B = -\frac{a}{b}A = \frac{rC}{\sqrt{4LC - r^2C^2}}EC$$

and

$$\begin{aligned} q &= EC + \epsilon^{at}\sqrt{A^2 + B^2}\sin\left(bt + \tan^{-1}\frac{A}{B}\right) \\ &= EC + \epsilon^{at}A\sqrt{1 + \frac{a^2}{b^2}}\sin\left\{bt + \tan^{-1}\left(-\frac{b}{a}\right)\right\} \\ &= EC - EC\epsilon^{at}\sqrt{1 + \frac{a^2}{b^2}}\sin\left\{bt + \tan^{-1}\left(-\frac{b}{a}\right)\right\} \\ &= EC - EC\frac{2\sqrt{LC}}{\sqrt{4LC - r^2C^2}}\epsilon^{-\frac{rt}{2L}}\sin\left\{\frac{\sqrt{4LC - r^2C^2}}{2LC}t \right. \\ &\qquad\qquad \left. + \tan^{-1}\frac{\sqrt{4LC - r^2C^2}}{rC}\right\} \qquad (101) \end{aligned}$$

From equation (100), the expression for the current becomes

$$i = \frac{2EC}{\sqrt{4LC - r^2C^2}}\epsilon^{-\frac{rt}{2L}}\sin\frac{\sqrt{4LC - r^2C^2}}{2LC}t \qquad (102)$$

It is evident that the charge and the current have the same initial and final values as in Case IA, just discussed. There is, however, an oscillation about these values, the amplitude of the oscillations decreasing logarithmically. The period of these

oscillations is determined by the fact that, when $\dfrac{\sqrt{4LC - r^2C^2}}{2LC}\,t$ increases by 2π, the charge and current pass through one complete cycle of values. Hence the increase in time t' is given by

$$t'\,\frac{\sqrt{4LC - r^2C^2}}{2LC} = 2\pi$$

Therefore,

$$T = t' = \frac{2\pi\sqrt{LC}}{\sqrt{1 - \dfrac{r^2C}{4L}}} \qquad (103)$$

If r^2C is very small compared to $4L$, so that $\dfrac{r^2C}{4L}$ may be neglected, T, the time of a complete period, is

$$T = 2\pi\sqrt{LC} \qquad (104)$$

As $\dfrac{r^2C}{4L}$ approaches unity, the periodic time T approaches infinity. [See equation (103).]

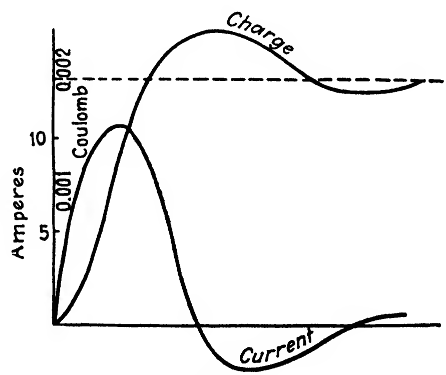


Fig. 47.

Figure 47 shows equations (101) and (102) plotted for the following constants:

$E = 2000$ volts	$L = 0.0125$ henry
$r = 100$ ohms	$C = 1$ microfarad

CASE Ic. $r^2C = 4L$ or $r = 2\sqrt{\frac{L}{C}}$. In this case the roots a_1 and a_2, equations (87) and (88), page 162, are equal; *i.e.*, $a_1 = a_2 = a = -\frac{r}{2L}$. Equations (89) and (90) now become

$$q = A_1 \epsilon^{at} + A_2 t\epsilon^{at} + EC^* \tag{105}$$

and

$$i = A_1 a \epsilon^{at} + A_2 at\epsilon^{at} + A_2 \epsilon^{at} \tag{106}$$

If $t = 0$, $q = 0$ and $i = 0$. Putting these values in equations (105) and (106) gives

$$A_1 = -EC$$

$$A_2 = aEC = -\frac{r}{2L}EC$$

and

$$q = EC - EC\left(1 + \frac{rt}{2L}\right)\epsilon^{-\frac{rt}{2L}} \tag{107}$$

$$i = \frac{E}{L}t\epsilon^{-\frac{rt}{2L}} \tag{108}$$

It is important to note that in this case, as in Case IA, both charge and current are non-oscillatory.

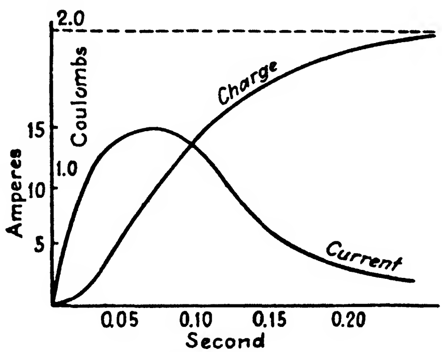


FIG. 48.

In Fig. 48 are shown graphically the curves for charge and current for the following constants:

$E = 2000$ volts $\qquad L = 2.5$ henrys

$r = 100$ ohms $\qquad C = 1000$ microfarads

* See Murray, Differential Equations.

Case II. Discharge of a Condenser through a Circuit Containing Constant Resistance and Constant Self-inductance in Series. (Figure 45, page 161, key on b.) The differential equation for this condition is

$$0 = ri + L\frac{di}{dt} + \frac{q}{C} \tag{109}$$

or

$$0 = r\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} \tag{110}$$

In this case, the final or steady state is zero charge, since the applied voltage is zero. The equation for charge contains only a transient term. The steady term u in equation (57), page 150, becomes zero since the final charge is zero.

$$q = A_1\epsilon^{a_1 t} + A_2\epsilon^{a_2 t} \tag{111}$$

Since $i = \frac{dq}{dt}$,

$$i = A_1 a_1 \epsilon^{a_1 t} + A_2 a_2 \epsilon^{a_2 t} \tag{112}$$

The roots a_1 and a_2 have the same values as in Case I, equations (87) and (88), page 162. The method of solving equations (111) and (112) is the same as was used for finding the transient terms of equations (89) and (90) under Case I, pages 162 and 163. As before, there are three cases to consider, according as r^2C is greater than $4L$, less than $4L$ or equal to $4L$.

Case IIa. $r^2C > 4L$ or $r > 2\sqrt{\frac{L}{C}}$. In this case, the roots a_1 and a_2 are essentially negative and have the same values as in Case Ia, page 163.

$$q = EC\left\{\frac{rC + \sqrt{r^2C^2 - 4LC}}{2\sqrt{r^2C^2 - 4LC}}\epsilon^{-\left(\frac{rC - \sqrt{r^2C^2 - 4LC}}{2LC}\right)t} - \frac{rC - \sqrt{r^2C^2 - 4LC}}{2\sqrt{r^2C^2 - 4LC}}\epsilon^{-\left(\frac{rC + \sqrt{r^2C^2 - 4LC}}{2LC}\right)t}\right\} \tag{113}$$

and

$$i = \frac{-EC}{\sqrt{r^2C^2 - 4LC}}\left\{\epsilon^{-\left(\frac{rC - \sqrt{r^2C^2 - 4LC}}{2LC}\right)t} - \epsilon^{-\left(\frac{rC + \sqrt{r^2C^2 - 4LC}}{2LC}\right)t}\right\} \tag{114}$$

In Fig. 49 are shown graphically the curves of charge and current during the discharge of a condenser through a circuit

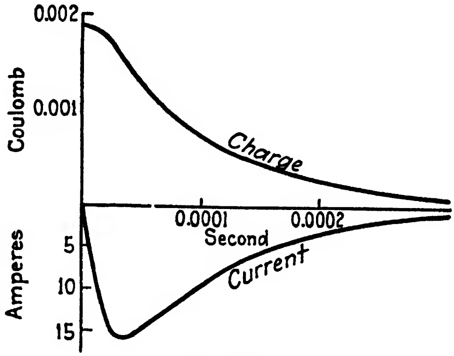


FIG. 49.

having constant resistance and constant self-inductance in series for the following constants:

$$E = 2000 \text{ volts} \qquad L = 0.0016 \text{ henry}$$
$$r = 100 \text{ ohms} \qquad C = 1 \text{ microfarad}$$

CASE IIB. $r^2C < 4L$ or $r < 2\sqrt{\frac{L}{C}}$. In this case, as in Case IB, page 164, the roots a_1 and a_2 are imaginary and have the same values as in that case. Equations (111) and (112) become

$$q = EC\frac{2\sqrt{LC}}{\sqrt{4LC - r^2C^2}}\epsilon^{-\frac{rt}{2L}}\sin\left(\frac{\sqrt{4LC - r^2C^2}}{2LC}t + \tan^{-1}\frac{\sqrt{4LC - r^2C^2}}{rC}\right) \quad (115)$$

$$i = \frac{-2EC}{\sqrt{4LC - r^2C^2}}\epsilon^{-\frac{rt}{2L}}\sin\frac{\sqrt{4LC - r^2C^2}}{2LC}t \quad (116)$$

The charge and current have the same initial and the same final values as in Case IIA. They oscillate about these final values with an amplitude which decreases logarithmically.

The time of one complete vibration for charge or current is the same as in Case IB and is found in the same way.

$$T = \frac{2\pi\sqrt{LC}}{\sqrt{1 - \frac{r^2C}{4L}}} \qquad f = \frac{1}{T} = \frac{\sqrt{1 - \frac{r^2C}{4L}}}{2\pi\sqrt{LC}} \tag{117}$$

If r^2C is negligibly small as compared to $4L$,

$$T = 2\pi\sqrt{LC} \qquad f = \frac{1}{2\pi\sqrt{LC}}$$

If $r^2C = 4L$, T, the time of a complete vibration, is infinite.

The oscillatory discharge of a condenser is shown graphically in Fig. 50 for the following constants:

$$E = 2000 \text{ volts} \qquad L = 0.0125 \text{ henry}$$
$$r = 100 \text{ ohms} \qquad C = 1 \text{ microfarad}$$

CASE IIc. $r^2C = 4L$ or $r = 2\sqrt{\frac{L}{C}}$. Here the roots a_1 and a_2 are equal as in Case Ic, page 167, and have the same values as in that case. The method of solving the differential equation

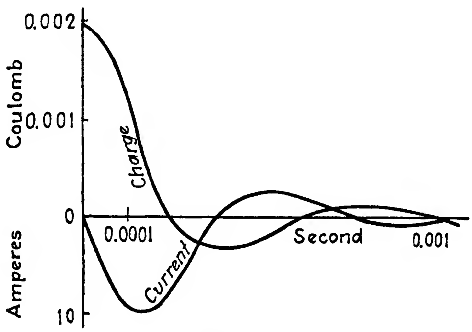


FIG. 50.

is the same as in Case Ic, except that the term EC, which represents the steady condition, is zero. The solution involves nothing but the transient term, since the final charge must be zero. From equations (105) and (106), page 167,

$$q = A_1\epsilon^{at} + A_2t\epsilon^{at} \tag{118}$$

and

$$i = A_1a\epsilon^{at} + A_2at\epsilon^{at} + A_2\epsilon^{at} \tag{119}$$

The solution of equations (118) and (119) gives

$$q = EC\left(1 + \frac{r}{2L}t\right)\epsilon^{-\frac{rt}{2L}} \tag{120}$$

$$i = -\frac{E}{L}t\epsilon^{-\frac{rt}{2L}} \tag{121}$$

where $EC = Q$ and E are, respectively, the initial charge and initial voltage of the condenser. Equations (120) and (121) are shown graphically in Fig. 51 for the following constants:

$E = 2000$ volts $\qquad L = 2.5$ henrys
$r = 100$ ohms $\qquad C = 1000$ microfarads

CASE III. A SIMPLE HARMONIC ELECTROMOTIVE FORCE IMPRESSED ON A CIRCUIT CONTAINING CONSTANT RESISTANCE, CONSTANT SELF-INDUCTANCE AND CONSTANT CAPACITANCE IN SERIES. In this case, the impressed electromotive force is of the form

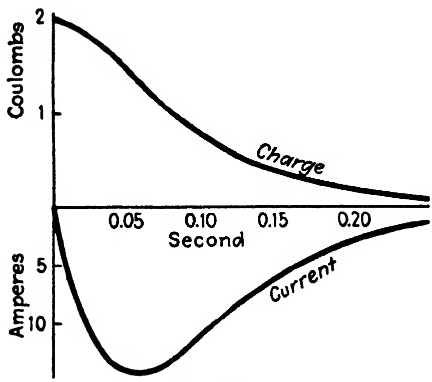


FIG. 51

$$e = E_m \sin(\omega t + \alpha)$$

and

$$E_m \sin(\omega t + \alpha) = ri + L\frac{di}{dt} + \frac{q}{C} \tag{122}$$

or

$$E_m \sin(\omega t + \alpha) = r\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} \tag{123}$$

The complete solution of this linear differential equation of the second order, the first term of which is a function of t, is the sum of the transient and the steady value of the charge. As in all preceding cases, the solution is of the form

$$q = Y + u$$

where Y is the transient term, *i.e.*, the complementary function of the differential equation, and u is the particular integral and represents the steady state. The transient term or complementary function is found, as in all other cases, by putting the first term, $E_m \sin(\omega t + \alpha)$, equal to zero. The value of the complementary function evidently depends on the relation of r^2C to $4L$. Its value is of the same form as in Cases IIA, B and C. It is

$$q = A_1\epsilon^{a_1t} + A_2\epsilon^{a_2t} \tag{124}$$

$$i = A_1a_1\epsilon^{a_1t} + A_2a_2\epsilon^{a_2t} \tag{125}$$

It is oscillatory when r^2C is less than $4L$, or r is less than $2\sqrt{\frac{L}{C}}$, the frequency being given by equation (117), page 170, and is

$$f = \frac{1}{T} = \frac{\sqrt{1 - \frac{r^2C}{4L}}}{2\pi\sqrt{LC}} \tag{126}$$

which is determined by the constants of the circuit. The frequency is entirely independent of the frequency of the impressed voltage. The constants for the transient terms for charge and current when the impressed voltage is sinusoidal are determined from the charge and the current when $t = 0$. When the transient terms are oscillatory, the constants are determined from equations (99) and (100), page 165.

The transient is of importance in all switching operations on circuits containing resistance, self-inductance and capacitance in series, since dangerous oscillations may be produced if the resistance of the circuit is low compared with $2\sqrt{\frac{L}{C}}$. It is important in all radio work involving tuning of series circuits. Most radio circuits have mutual inductance as well as self-

inductance, and for such circuits the conditions are not so simple as those just discussed.

The transients in transmission lines may be of great importance during switching operations, short circuits etc., but the equations just developed do not apply to a transmission line, since a transmission line is not a simple series circuit. It is a circuit containing series resistance and series inductance, but it has parallel capacitance and leakance between the conductors and between the conductors and the earth.

The particular integral, *i.e.*, the term representing the steady state for equation (123), page 171, may be found most easily by a method similar to that used in Case III for a circuit containing constant resistance and constant self-inductance in series and also in Case III for a circuit containing constant resistance and constant capacitance in series.

Under steady conditions, the voltage drop across the circuit is made up of three parts, viz., $ri = r\frac{dq}{dt}$, the voltage drop across the resistance, $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$, the voltage drop across the self-inductance, and $\frac{q}{C} = \frac{1}{C}\int i\,dt$, the voltage drop across the capacitance. If r, L and C are constant, the reasoning that was used in Case III for a circuit containing constant resistance and constant self-inductance in series and also in Case III for a circuit containing constant resistance and constant capacitance in series shows that both the charge and current must vary sinusoidally with time when the impressed electromotive force is sinusoidal.

Assume that the charge is given by

$$q = Q_m \sin(\omega t + \alpha + \theta') \tag{127}$$

where θ' is the phase angle of the charge with respect to the impressed electromotive force. Then,

$$\begin{aligned} E_m \sin(\omega t + \alpha) &= r\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} \\ &= \omega r Q_m \cos(\omega t + \alpha + \theta') \\ &\qquad - \omega^2 L Q_m \sin(\omega t + \alpha + \theta') \\ &\qquad + \frac{1}{C} Q_m \sin(\omega t + \alpha + \theta') \end{aligned} \tag{128}$$

Since the cosine of an angle is equal to the sine of ninety degrees plus the angle, equation (128) may be written in the following form:

$$E_m \sin(\omega t + \alpha) = \omega r Q_m \sin(\omega t + \alpha + \theta' + 90°) - \omega^2 L Q_m \sin(\omega t + \alpha + \theta') + \frac{1}{C} Q_m \sin(\omega t + \alpha + \theta') \qquad (129)$$

The vectors corresponding to the terms of equation (129) are plotted in Fig. 52 for the instant of time $t = -\frac{\alpha + \theta'}{\omega}$. This instant of time was chosen because it puts the vector representing the charge along the axis of reference.

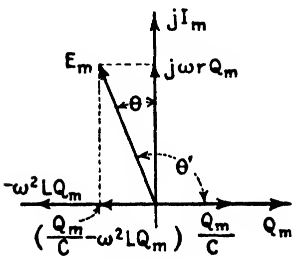


FIG. 52.

From Fig. 52 it is obvious that

$$E_m = \sqrt{(\omega r Q_m)^2 + \left(\frac{1}{C} Q_m - \omega^2 L Q_m\right)^2}$$
$$= \omega Q_m \sqrt{r^2 + \left(\frac{1}{\omega C} - \omega L\right)^2} \qquad (130)$$

and

$$Q_m = \frac{E_m}{\omega \sqrt{r^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \qquad (131)$$

$$\tan \theta' = \frac{\omega r Q_m}{\frac{1}{C} Q_m - \omega^2 L Q_m}$$
$$= \frac{r}{\frac{1}{\omega C} - \omega L} \qquad (132)$$

The charge lags the voltage $E_m \sin(\omega t + \alpha)$, impressed across the circuit, by the angle θ'. The angle θ', therefore, represents the lag of the charge behind the impressed voltage or the lead of the impressed voltage with respect to the charge. When $\frac{1}{\omega C}$ is greater than ωL, $\tan \theta'$ is positive and θ' is less than 90 degrees. When $\frac{1}{\omega C}$ is less than ωL, $\tan \theta'$ is negative and the

charge lags the impressed voltage by more than 90 degrees but less than 180 degrees. If r is zero, θ' is also zero. In this case the charge is in phase with or in opposition to the voltage impressed across the circuit according as $\omega^2 Q_m$ is less or greater than $\frac{Q_m}{C}$.

The expression for the charge when steady conditions have been reached is, therefore,

$$q = \frac{E_m}{\omega\sqrt{r^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \sin\left\{\omega t + \alpha - \tan^{-1}\frac{r}{\left(\frac{1}{\omega C} - \omega L\right)}\right\} \qquad (133)$$

The angle $\theta' = \tan^{-1}\frac{r}{\left(\frac{1}{\omega C} - \omega L\right)}$ in equation (133) is a lag angle, since equation (133) is an equation for the charge in terms of the impressed electromotive force. The charge lags the impressed electromotive force $E_m \sin(\omega t + \alpha)$ by the angle θ'.

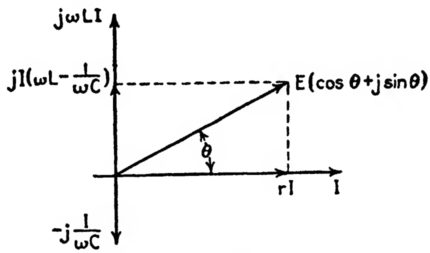


FIG. 53.

Since $i = \frac{dq}{dt}$, the current under steady conditions is

$$i = \frac{E_m}{\sqrt{r^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left\{\omega t + \alpha - \tan^{-1}\frac{r}{\left(\frac{1}{\omega C} - \omega L\right)}\right\}$$

$$= \frac{E_m}{\sqrt{r^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \sin\left\{\omega t + \alpha - \tan^{-1}\frac{r}{\left(\frac{1}{\omega C} - \omega L\right)} + 90^\circ\right\} \qquad (134)$$

The current, therefore, leads the charge by 90 degrees. The vector representing the current is plotted in Fig. 53.

Let θ be the angle which the current makes with the impressed electromotive force E_m. Then,

$$\theta = \theta' - 90^\circ$$

$$-\theta = \tan^{-1}\frac{\left(\frac{1}{C}Q_m - \omega^2 L Q_m\right)}{\omega r Q_m}$$

$$= -\tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{r}$$

Equation (134) may therefore be written

$$i = \frac{E}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left\{\omega t + \alpha - \tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{r}\right\} \quad (135)$$

The signs of ωL and $\frac{1}{\omega C}$ in the radical $\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$, of equation (135), are reversed in the expression $\left(\omega L - \frac{1}{\omega C}\right)$ to make it correspond to the like term in the expression for the tangent of the angle θ. This can be done since the term $\left(\omega L - \frac{1}{\omega C}\right)$ is squared and changing its sign does not alter its squared value.

The complete solution of equation (123), page 171, for charge is given by adding equation (133) to equation (124), page 172

$$q = Y + u$$
$$= A_1\epsilon^{a_1 t} + A_2\epsilon^{a_2 t}$$
$$+ \frac{E_m}{\omega\sqrt{r^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \sin\left\{\omega t + \alpha - \tan^{-1}\frac{r}{\left(\omega C - \frac{1}{\omega L}\right)}\right\} \quad (136)$$

The complete solution of equation (123), page 171, for current, is given by adding equation (135) to equation (125), page 172.

$$
\begin{aligned}
i &= Y + u \\
&= A_1 a_1 \epsilon^{a_1 t} + A_2 a_2 \epsilon^{a_2 t} \\
&+ \frac{E_m}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left\{\omega t + \alpha - \tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{r}\right\} \qquad (137)
\end{aligned}
$$

After a brief interval of time, the transient terms in equations (136) and (137) become sensibly zero for circuits with constants ordinarily met in practice and the charge and current then become simple harmonic functions of time.

The maximum value of the current under steady conditions is found by dividing the maximum value of the impressed electromotive force by $\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$. The current lags the voltage by an angle whose tangent is $\frac{\omega L - \frac{1}{\omega C}}{r}$. When $\frac{1}{\omega C}$ is greater than ωL, the angle θ becomes negative and is then equivalent to an angle of lead. In this case, the current actually leads the electromotive force impressed on the circuit by an angle θ. When $\frac{1}{\omega C}$ is less than ωL, the angle θ is positive and is actually an angle of lag. In this case, the current actually lags the voltage impressed on the circuit.

Since the effective value of a sinusoidal wave is equal to its maximum value divided by the square root of two, it is evident that the effective value of the current is found by dividing the effective value of the voltage by $\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$. This current leads or lags the impressed voltage according as $\frac{1}{\omega C}$ is greater or less than ωL

$$
I = \frac{E}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)}} = \frac{E}{z} \qquad (138)
$$

$$\theta = \tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{r} \tag{139}$$

Impedance and Reactance of a Series Circuit Containing Constant r, Constant L and Constant C.—The quantity

$$z = \sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{140}$$

is called the *impedance* and is measured in ohms. It is constant only when resistance, self-inductance, capacitance and frequency are constant. It may increase or decrease with an increase in frequency depending on the relative values of ωL and $\frac{1}{\omega C}$. The expression $\omega L = x_L$ is the inductive reactance and the expression $-\frac{1}{\omega C} = x_C$ is the capacitive reactance. $\left(\omega L - \frac{1}{\omega C}\right) = x_L + x_C = x_0$ is the resultant reactance. It should be noted that inductive reactance is always positive but capacitive reactance is always negative. The resultant reactance may be either positive or negative, depending upon the relative values of x_L and x_C. Inductive reactance always increases with increase of frequency. Capacitive reactance always decreases with increase of frequency.

When $\omega L = \frac{1}{\omega C}$, $x_0 = 0$ and the current is given by $\frac{E}{r}$ and is in phase with the impressed electromotive force. Under this condition the circuit is said to be in resonance. Resonance will be considered in detail later.

Vector Method of Determining the Steady Component of the Current in a Circuit Containing Constant Resistance, Constant Self-inductance and Constant Capacitance in Series.—The current vector, $\bar{I}$ amperes effective, will be taken along the axis of reals. The effective values of the active and reactive components of the impressed electromotive force are, respectively, $r\bar{I}$ and $j\bar{I}(x_L + x_C) = j\bar{I}\left(\omega L - \frac{1}{\omega C}\right)$. $j\omega L\bar{I} = j\bar{I}x_L$ is the reactive component due to the self-inductance. $-j\frac{\bar{I}}{\omega C} = j\bar{I}x_C$ is the

reactive component due to the capacitance. $j\omega L\bar{I}$ leads the current and $-j\frac{\bar{I}}{\omega C}$ lags the current.

$$\bar{E} = r\bar{I} + j(x_L + x_C)\bar{I} = \bar{I}[r + j(x_L + x_C)] = \bar{I}\bar{z} \quad (141)$$

$$\bar{I} = \frac{\bar{E}}{r + j(x_L + x_C)} = \frac{\bar{E}}{r + j\left(\omega L - \frac{1}{\omega C}\right)} \quad (142)$$

When substituting numerical values it must be remembered that x_C is $\frac{-1}{\omega C}$ and is negative.

The vectors, corresponding to equation (141), are plotted in Fig. 53, page 175.

The complex expression for the impedance is, therefore,

$$\bar{z} = r + j(x_L + x_C) = r + j\left(\omega L - \frac{1}{\omega C}\right)$$

The magnitude of z in ohms is $\sqrt{r^2 + (x_L + x_C)^2}$.

Polar Expression for the Impedance of a Circuit Containing Constant Resistance, Constant Self-inductance and Constant Capacitance in Series.—The polar expression for the impedance of a circuit containing constant resistance, constant self-inductance and constant capacitance in series is (see pages 145 and 160)

$$\bar{z} = z\underline{|\theta} \quad (143)$$

where the angle θ is determined by the relation $\theta = \tan^{-1}\frac{x_L + x_C}{r} = \tan^{-1}\frac{x_0}{r}$. If the inductive reactance predominates, θ is positive, but if the capacitive reactance predominates, θ is negative.

Impedance may be written in four different ways, namely:

$$\begin{aligned}\bar{z} &= r + jx_0 \\ &= z(\cos\theta + j\sin\theta) \\ &= z\underline{|\theta} \\ &= z\epsilon^{j\theta}\end{aligned}$$

Equation for the Velocity and Displacement of a Mechanical System Having Friction, Mass and Elasticity.—Before leaving the consideration of circuits containing constant resistance,

constant self-inductance and constant capacitance in series, it will be of interest to compare the equations developed for such circuits with the equations for displacement and velocity of a mechanical system having constant friction, constant mass and constant elasticity. It has already been pointed out that self-inductance and capacitance are analogous to mass and elasticity, respectively, in mechanics. Electrical resistance corresponds to mechanical friction. Current corresponds to velocity and charge to displacement in a mechanical system.

Consider a torsional pendulum. Let $\mathcal{J}$ be the moment of inertia of the pendulum about its axis of oscillation. Also let $\mathcal{L}$ and $\mathcal{G}$ be, respectively, the length of the torsion wire and its coefficient of torsional rigidity. The radius of the torsion wire is r. Then if ω and α are, respectively, the angular velocity and the angular displacement of the pendulum from its mean position due to an applied couple M_0,

$$\text{Reaction due to friction} = M_f = k_f\omega$$

$$\text{Reaction due to inertia} = M_i = \mathcal{J}\frac{d\omega}{dt}$$

$$\text{Reaction due to elasticity} = M_e = \mathcal{G}\frac{\pi r^4}{2}\frac{\alpha}{\mathcal{L}}$$

Since the displacing couple M_0 must balance the sum of the couples due to the reactions caused by the motion of the pendulum,

$$\begin{aligned} M_0 &= M_f + M_i + M_e \\ &= k_f\omega + \mathcal{J}\frac{d\omega}{dt} + \mathcal{G}\frac{\pi r^4}{2}\frac{\alpha}{\mathcal{L}} \\ &= k_f\omega + k_i\frac{d\omega}{dt} + \frac{1}{k_e}\alpha \end{aligned} \tag{144}$$

Since $\omega = \frac{d\alpha}{dt}$, equation (144) may be written

$$M_0 = k_f\frac{d\alpha}{dt} + k_i\frac{d^2\alpha}{dt^2} + \frac{1}{k_e}\alpha \tag{145}$$

Equation (145) corresponds exactly to equation (83), page 162, for an electrical circuit containing constant resistance, constant self-inductance and constant capacitance in series, and its solution takes the same form as the solution for that equation.

When M_0 is zero, *i.e.*, when the pendulum oscillates freely without applied accelerating or retarding couples, equation (145) becomes

$$0 = k_f\frac{d\alpha}{dt} + k_i\frac{d^2\alpha}{dt^2} + \frac{1}{k_e}\alpha \tag{146}$$

Equation (146) corresponds to equation (110), page 168, for the discharge of an electrical circuit containing constant resistance, constant self-inductance and constant capacitance in series. The solution of equation (146) takes three forms corresponding to cases *a*, *b* and *c* for equation (110).

Case	Electrical circuit	Mechanical system
(*a*)	$r > 2\sqrt{\frac{L}{C}}$ Charge Q is a maximum when $t = 0$. It then decreases, approaching zero as a limit. Current starts at zero when $t = 0$, rises to a maximum, then decreases, approaching zero as a limit. There is no oscillation.	$k_f > 2\sqrt{\frac{k_i}{k_e}}$ Displacement α is a maximum when $t = 0$. It then decreases, approaching zero as a limit. Angular velocity ω starts at zero when $t = 0$, rises to a maximum, then decreases, approaching zero as a limit. There is no oscillation.
(*b*)	$r < 2\sqrt{\frac{L}{C}}$ Charge and current have the same initial and final values as in Case (*a*). They oscillate about these values with amplitudes which decrease logarithmically.	$k_f < 2\sqrt{\frac{k_i}{k_e}}$ Displacement and velocity have the same initial and final values as in Case (*a*). They oscillate about these values with amplitudes which decrease logarithmically.
(*c*)	$r = 2\sqrt{\frac{L}{C}}$ Charge and current have the same initial and final values as in Cases (*a*) and (*b*). There is no oscillation.	$k_f = 2\sqrt{\frac{k_i}{k_e}}$ Displacement and velocity have the same initial and final values as in Cases (*a*) and (*b*). There is no oscillation.

When $k_f < 2\sqrt{\frac{k_i}{k_e}}$, there is oscillation, *i.e.*, if the pendulum is displaced, it comes to rest after a series of oscillations about its

mean position. These oscillations decrease logarithmically with time. The period of vibration is

$$T = \frac{2\pi\sqrt{k_i k_e}}{\sqrt{1 - \frac{k_f^2 k_e}{4k_i}}}$$

When k_f is small,

$$T = 2\pi\sqrt{k_i k_e}$$

Although the resistance of an electrical circuit may frequently be too high for electrical oscillations to take place, the friction of a pendulum is seldom too high to prevent oscillation unless the pendulum is specially damped. The vibrating element of an oscillograph is an example of a highly damped torsional pendulum. If it is displaced, it comes to rest without oscillation.

CHAPTER VI

MUTUAL INDUCTION, COUPLED CIRCUITS AND AIR-CORE TRANSFORMER

Mutual Induction.—Up to the present point, the only reactions which have been considered, when a current in a circuit is varied, are those existing or arising in the circuit itself. The equations which have been developed hold so long as the circuit considered is not in the neighborhood of other circuits or is so related to any other circuit in its vicinity that there can be no interaction between them. In general, however, when two or more circuits are in proximity, any change in the current in one of them causes inductive effects in the others.

Consider two circuits which are in proximity to each other. When one of two such circuits carries current, a magnetic field is established. This field links not only the circuit by which it is produced, but, in general, a certain portion of it also links the other circuit. The relative amounts of flux linking each circuit depend chiefly on the relative sizes and shapes of the circuits, their relative positions and the permeability of the surrounding medium. All of the flux produced by one circuit can never link the other, although the difference between the flux linking one and that linking the other may be made very small by closely interwinding them. On the other hand, the difference may be very great when the circuits are far removed from each other or are placed with the axis of one in the plane of the other.

If each circuit is carrying current, any change in the current of either is accompanied by a change in the flux linkages of each circuit. The change in the flux linkages of each circuit induces in each a voltage which is equal to the time rate of change of flux linkages for the circuit. The rate of change of flux linkages of one circuit, with respect to change in current in the other, forms one of the most important and fundamental constants in the electric theory of circuits.

If both circuits are now closed and carrying current, any change in the current in one induces in the other a voltage which causes a current, and, according to the law of Lenz, this second current must have such a direction as to oppose the change in the flux producing it. When the current in the first circuit increases, the current *induced* in the second circuit must be in a direction opposite to that in the first circuit. When the current in the first circuit decreases, the current *induced* in the second circuit must be in the same direction as that in the first circuit. No change can take place in the current of one without producing a corresponding change by induction in the current of the other. Any change in the current of one results in a current in the other, since there is mutual induction between the two.

Mutual induction plays an important part in the operation of most alternating-current apparatus. The operation of certain types of apparatus, such as the transformer and the induction motor, depends entirely upon it. Without the transformer and the induction motor, the present development of alternating-current systems of power distribution and utilization would be impossible. Certain types of coupled circuits largely used in both wire and wireless communication circuits also depend upon mutual inductance.

Coefficient of Mutual Induction or Mutual Inductance.—The self-inductance of a circuit or its coefficient of self-induction was defined as the rate of change in the flux linkages of the circuit with respect to its current. When the permeability of the surrounding medium is constant, the self-inductance of the circuit is constant. When the permeability varies with current strength, the self-inductance also varies with current strength. When the permeability is constant, self-inductance may be defined as the change in the flux linkages of the circuit per unit change in its current. These definitions of self-inductance assume that no change is produced in the currents in other circuits in the neighborhood. When the current is varied in a circuit having a self-inductance *L*, a voltage of self-induction is induced in it which is equal to the time rate of change of the flux linkages of the circuit. The voltage drop due to self-inductance (assumed constant) is

$$e_{1L} = L_1 \frac{di_1}{dt} \tag{1}$$

Similarly, the mutual inductance or coefficient of mutual induction of two circuits is defined as the rate of change in the flux linkages of one with respect to the current in the other. When the permeability of the circuits is constant, their mutual inductance is constant. When the permeability varies with the current strength in either circuit, the mutual inductance also varies with current strength. When the permeability is constant, the mutual inductance of the two circuits may be defined as the change of flux linkages for either per unit change of current in the other. When the current is varied in one of the two circuits having a mutual inductance M, a voltage of mutual induction is induced in the other and is equal to the time rate of change of the flux linkages for that circuit. The voltage drop in circuit 2, due to the mutual inductance (assumed constant) of circuit 1 on circuit 2, is

$$e_{2M} = M_{12}\frac{di_1}{dt} \tag{2}$$

Self-inductance and mutual inductance are both measured in the same unit, the henry. The mutual inductance of two circuits is one henry when the rate of change of flux linkages of either, with respect to the current in amperes in the other, is 10^8 flux linkages per ampere. When the permeability is constant, two circuits have a mutual inductance of one henry when a change of 10^8 flux linkages is produced in either by a change of one ampere in the current of the other. These definitions assume that the current in one circuit is constant while the current in the other circuit is changed. When the coefficient of self-induction or mutual induction is expressed in henrys and the rate of change of current is in amperes per second, the induced voltage is in volts.

If the permeability of a circuit is constant and there is no magnetic leakage between its turns, *i.e.*, if all the flux produced by each turn links all the turns, the coefficient of self-induction is

$$L_1 = \frac{4\pi N_1}{\mathcal{R}}N_1 \text{ abhenrys} \tag{3}$$

where N_1 is the number of turns and $\mathcal{R}$ the reluctance of the magnetic circuit.

If N_2 is the number of turns in a second circuit so related to the first that all the flux produced by the first circuit links

all of the turns of the second circuit, the coefficient of mutual induction of the first circuit on the second circuit is

$$M_{12} = \frac{4\pi N_1}{\mathcal{R}} N_2 \text{ abhenrys} \tag{4}$$

All the flux produced by the first circuit can never link all the turns of the second circuit, and all the flux produced by the second circuit can never link all the turns of the first circuit. There is always some leakage of magnetic flux between the two circuits. By closely interwinding the two circuits, the magnetic leakage may be made very small. It is, however, often very great.

Although it is possible to calculate the self-inductance and mutual inductance of certain very simple circuits with a fair degree of accuracy, accurate calculation of either self-inductance or mutual inductance is usually impossible.

Voltage Drop across Circuits Having Resistance, Self-inductance and Mutual Inductance.—From the definitions of self-inductance and mutual inductance, it is obvious that the voltage drops across two circuits having resistance, self-inductance and mutual inductance are given by the following differential equations:

$$v_1 = r_1 i_1 + L_1 \frac{di_1}{dt} + M_{21} \frac{di_2}{dt} \tag{5}$$

$$v_2 = r_2 i_2 + L_2 \frac{di_2}{dt} + M_{12} \frac{di_1}{dt} \tag{6}$$

When no magnetic material is present, the coefficients of self-induction and mutual induction in equations (5) and (6) are constant. When the flux produced by the currents i_1 and i_2 is in magnetic material, the coefficients of self-induction and mutual induction vary with the currents. They are complicated functions of the currents. In general, when magnetic material is present, no exact solution of equations (5) and (6) can be obtained.

It will be shown that the coefficient of mutual induction of two circuits, in the absence of magnetic material, is the same whether it is taken for circuit 1 with respect to circuit 2 or for circuit 2 with respect to circuit 1. Under this condition the subscripts on the coefficient of mutual induction, M, between

two circuits, have no significance and may be omitted. Subscripts are necessary when dealing with more than two circuits in order to indicate which two circuits are considered.

The Coefficients of Mutual Induction, M_{12} and M_{21}, of Two Circuits in a Medium of Constant Permeability Are Equal.—Let the permeability of the medium be constant. This assumes that there is no magnetic material present. The coefficients of mutual induction, M_{12} of circuit 1 on circuit 2, and M_{21} of circuit 2 on circuit 1, are equal. This statement is true without regard to the sizes, shapes or positions of the two circuits.

The total electromagnetic energy in an electrical system consisting of two circuits, 1 and 2, which have constant self-inductance and mutual inductance and carry currents I_1 and I_2, respectively, is obviously independent of the order or manner in which the currents are established. This follows from the law of conservation of energy.

Let I_1 and I_2 be the final values of the currents in the circuits 1 and 2, respectively. Consider I_2 to be zero and establish I_1. The energy due to establishing I_1 is due entirely to the self-inductance of circuit 1. While I_1 is increasing, a voltage is induced in circuit 2 by the mutual inductance of circuit 1 on circuit 2. No work can be done in circuit 2 by this voltage, since I_2 is zero. The total electromagnetic energy due to establishing the current I_1 is

$$\begin{aligned} W_1 &= \int_0^{I_1} i_1 L_1 \frac{di_1}{dt} dt \\ &= \frac{1}{2} L_1 I_1^2 \qquad (7) \end{aligned}$$

Now consider the current I_1 to be maintained constant while the current I_2 is established. The change produced in the electromagnetic energy of the system by the establishment of I_2 is made up of two parts, one due to the effect on I_2 of the voltage induced in circuit 2, the other due to the effect on I_1 of the voltage induced in circuit 1. The first part is due to the self-inductance of circuit 2, and the second part is due to the mutual inductance of circuit 2 on circuit 1. The change produced in the electromagnetic energy of the system by the establishment of I_2 is

$$W_2 = \int_0^{I_2} i_2 L_2 \frac{di_2}{dt} dt + \int_0^{I_2} I_1 M_{21} \frac{di_2}{dt} dt$$
$$= \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2 \tag{8}$$

The total change produced in the electromagnetic energy of the system by the establishment of the currents I_1 and I_2 is, therefore,

$$W_0 = W_1 + W_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2 \tag{9}$$

If I_2 had been established first, the energy equation would have been

$$W_0' = W_2 + W_1 = \frac{1}{2} L_2 I_2^2 + \frac{1}{2} L_1 I_1^2 + M_{12} I_2 I_1 \tag{10}$$

Since, according to the law of the conservation of energy, W_0 and W_0' must be equal, M_{12} and M_{21} must also be equal. In other words, the mutual inductance of circuit 1 with respect to circuit 2 is equal to the mutual inductance of circuit 2 with respect to circuit 1. In general, when there is no magnetic material present, the coefficient of mutual induction of two circuits is the same whether it is considered with respect to circuit 1 on circuit 2 or with respect to circuit 2 on circuit 1.

Magnetic Leakage and Leakage Coefficients.—Figure 54 shows two circuits having self-inductance and mutual inductance. All the flux that links circuit 1 does not link circuit 2 and all the flux that links circuit 2 does not link circuit 1. The flux that is common to both circuits, *i.e.*, the *mutual* flux, and the fluxes that link one circuit without linking the other, *i.e.*, the *leakage* fluxes, are indicated.

In Fig. 54, φ_M is the mutual flux. φ_{1S} and φ_{2S} are the leakage fluxes for circuits 1 and 2, respectively.

Let L_1 and L_2 be the self-inductances of the circuits 1 and 2, whose turns are N_1 and N_2, respectively. Assume that there is no magnetic material present. Under this condition, the self-inductances L_1 and L_2 are constant. Then, if all the flux produced in each turn of either circuit links all the turns of that circuit,

$$L_1 = \frac{4\pi N_1^2}{\mathcal{R}_1} \tag{11}$$

$$L_2 = \frac{4\pi N_2^2}{\mathcal{R}_2} \tag{12}$$

where $\mathcal{R}$ is the reluctance of the magnetic circuit.

Equations (11) and (12) would be approximately correct for solenoids which were long compared with their diameters.

It is obvious that all the flux produced by either circuit cannot link all the turns of the other. There must always be some

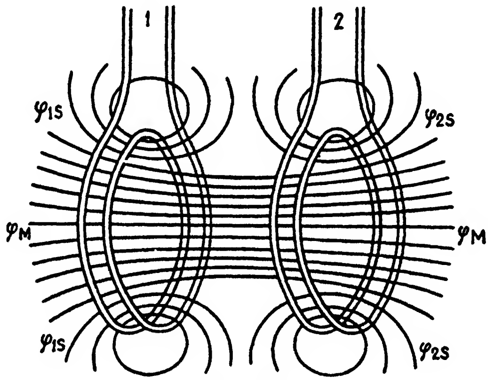


Fig. 54.

magnetic leakage between the two circuits, although, in certain cases, the leakage may be very small. On the other hand, it may be made very large by increasing the separation of the circuits and suitably changing their relative positions.

The magnetic leakage between two circuits is least when they are interwound, with the turns of one in close proximity to the turns of the other. The leakage would be zero if it were possible to make the corresponding turns of the two circuits occupy exactly the same position. The magnetic leakage is greatest when the two circuits are far apart and are placed so that the axis of one is in the plane of the other. In the last case, no flux produced by either circuit links the other and the mutual inductance of the circuits is zero.

The self-inductance of a circuit depends upon its size, shape, number of turns and the reluctance of its magnetic circuit. The mutual inductance of two circuits depends on their sizes, shapes,

numbers of turns, relative positions and the reluctance of the magnetic circuit. Magnetic leakage is determined by the relative sizes, shapes and positions of the circuits, the way the circuits are wound, *i.e.*, whether compact or spread out, and, if magnetic material is present, on the degree of saturation of the magnetic circuit. When an iron core is used, the leakage depends very largely on the position of the circuits on the core.

Let φ_1 be the total flux linking circuit 1 when it carries a current I_1, the current I_2 in circuit 2 being zero. Let φ_{1M} be the portion of φ_1 linking circuit 2. Then,

$$\frac{\varphi_1 - \varphi_{1M}}{\varphi_1} = k_1 \tag{13}$$

is known as the leakage coefficient of circuit 1 with respect to circuit 2.

Similarly, if φ_2 is the total flux linking circuit 2 when it carries a current I_2, the current I_1 being zero, and φ_{2M} is the portion of φ_2 linking circuit 1,

$$\frac{\varphi_2 - \varphi_{2M}}{\varphi_2} = k_2 \tag{14}$$

is the leakage coefficient of circuit 2 with respect to circuit 1.

The leakage coefficients of two circuits are not necessarily constant and independent of the current, and they are not necessarily equal. If the two circuits are of exactly the same shape and size and are symmetrically placed, their leakage coefficients are equal. The leakage coefficients are constant and independent of the current when there is no magnetic material in the path of the fluxes. They may be materially altered by the presence of magnetic material in the path of the mutual flux.

If k_1 and k_2 are the leakage coefficients of two circuits having mutual inductance and self-inductance,

$$M_{12} = \left\{\frac{L_1}{N_1}(1 - k_1)\right\}N_2 \tag{15}$$

$$M_{21} = \left\{\frac{L_2}{N_2}(1 - k_2)\right\}N_1 \tag{16}$$

It has already been shown that

$$M_{12} = M_{21} = M$$

when the permeability of the surrounding medium is constant.

Relation between the Mutual Inductance and Self-inductances of Two Circuits Containing No Magnetic Material, i.e., Having Constant Magnetic Reluctance.—Since M_{12} and M_{21} are equal, it follows from equations (15) and (16) that

$$M = M_{12} = M_{21} = \sqrt{L_1(1 - k_1) \times L_2(1 - k_2)} \qquad (17)$$

If there were no leakage, *i.e.*, if k_1 and k_2 were both zero, the mutual inductance of two circuits, whose self-inductances are L_1 and L_2, would be equal to $\sqrt{L_1 \times L_2}$.

Coupled Circuits.—It is often necessary to interconnect two or more circuits by means of a common unit which may be simple

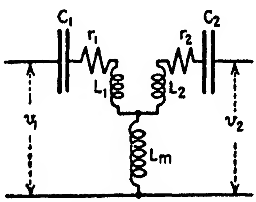


FIG. 55*a*.

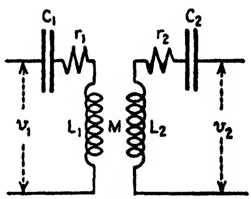


FIG. 55*b*.

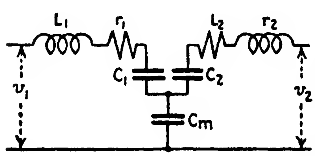


FIG. 55*c*.

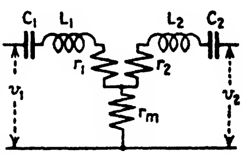


FIG. 55*d*.

inductance, mutual inductance, capacitance or resistance, or a combination of these. Such interconnected circuits are called *coupled circuits.* Simple circuits with the four kinds of coupling mentioned are shown in Fig. 55*a*, *b*, *c* and *d*.

Filters which are used in certain wire and wireless communication circuits are special types of coupled circuits or combinations of such circuits. They are used to pass freely certain frequencies and to suppress others. There are many types of filter circuits. The transformers used in the transmission of power and in com-

munication circuits are examples of circuits coupled by mutual *inductance.*

Coefficient of Coupling of Coupled Circuits.—The larger the fraction of inductance, reciprocal capacitance or resistance which is common to circuits that are coupled, the more they react on one another and the closer the coupling. The coefficient of coupling is the square root of the product of the fractions of inductance, reciprocal capacitance or resistance which are common to the circuits. The coefficients of coupling of the circuits shown in Fig. 55*a*, *b*, *c* and *d* are, respectively,

$$k_a = \sqrt{\left(\frac{L_m}{L_1 + L_m}\right)\left(\frac{L_m}{L_2 + L_m}\right)} \tag{18}$$

$$k_b = \sqrt{\left(\frac{M}{L_1}\right)\left(\frac{M}{L_2}\right)} \tag{19}$$

$$k_c = \sqrt{\left(\frac{C_1}{C_1 + C_m}\right)\left(\frac{C_2}{C_2 + C_m}\right)} \tag{20}$$

$$k_d = \sqrt{\left(\frac{r_m}{r_1 + r_m}\right)\left(\frac{r_m}{r_2 + r_m}\right)} \tag{21}$$

The coefficient of coupling of two circuits coupled by mutual inductance may be written in terms of their leakage coefficients by making use of equation (17),

$$\begin{aligned} k_b &= \sqrt{\frac{L_1(1 - k_1)L_2(1 - k_2)}{L_1 \times L_2}} \\ &= \sqrt{(1 - k_1)(1 - k_2)} \end{aligned} \tag{22}$$

The coupling between two circuits is said to be close when the coefficient of coupling is large. In radio work, circuits which have a coupling greater than 0.5 are said to be close-coupled. The degree of coupling that is desirable depends on the purpose for which the circuits are to be used. For commercial transformers with iron cores, the coupling between the primary and secondary windings is very close and may be as high as 0.98 or 0.99. Close coupling is generally desired in a commercial transformer, since the voltage regulation of a transformer depends very largely on the closeness of coupling between its primary and secondary windings. Good voltage regulation requires small magnetic leakage between primary and secondary windings and

therefore necessitates close coupling. Such close coupling as is used in commercial power transformers is undesirable in transformers *for radio work.*

General Equations for the Voltage Drops across Coupled Circuits.—The general equations for the voltage drops across the terminals of the circuits shown in Fig. 55*a*, *b*, *c* and *d* may easily be written. Call i_1 and i_2 the currents in the two circuits and call i_m the current in the common part of the two circuits in the case of the circuits shown in Fig. 55*a*, *b*, *c* and *d*. Then, for the circuit shown in Fig. 55*a*,

$$v_1 = \frac{1}{C_1}\int i_1 dt + r_1 i_1 + L_1\frac{di_1}{dt} + L_m\frac{di_m}{dt} \tag{23}$$

$$v_2 = \frac{1}{C_2}\int i_2 dt + r_2 i_2 + L_2\frac{di_2}{dt} + L_m\frac{di_m}{dt} \tag{24}$$

For the circuit shown in Fig. 55*b*,

$$v_1 = \frac{1}{C_1}\int i_1 dt + r_1 i_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \tag{25}$$

$$v_2 = \frac{1}{C_2}\int i_2 dt + r_2 i_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \tag{26}$$

Similar equations may be written for the circuits shown in Fig. 55*c* and *d*. If the constants of the circuits are independent of current strength, as when there is no magnetic material present, the equations for the voltages may be written in the vector form. For example, under the above conditions, equations (25) and (26) become

$$\bar{V}_1 = j\bar{I}_1\frac{-1}{\omega C_1} + \bar{I}_1 r_1 + j\bar{I}_1\omega L_1 + j\bar{I}_2\omega M \tag{27}$$

$$\bar{V}_2 = j\bar{I}_2\frac{-1}{\omega C_2} + \bar{I}_2 r_2 + j\bar{I}_2\omega L_2 + j\bar{I}_1\omega M \tag{28}$$

Equations (27) and (28) may be solved as simultaneous vector equations.

A special case of the solution of equations (25) and (26) occurs when the circuits have only resistance, self-inductance and mutual inductance, and the second circuit is short-circuited. This is equivalent to saying that the series capacitances C_1 and C_2 are infinite. In this case, equation (26) becomes

$$0 = r_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\frac{di_2}{dt} = -\left(\frac{M}{L_2}\frac{di_1}{dt} + \frac{r_2 i_2}{L_2}\right) \tag{29}$$

Substituting the value of $\frac{di_2}{dt}$ from equation (29) in equation (25) and remembering that C_1 is infinite,

$$v_1 = r_1 i_1 + \left(L_1 - \frac{M^2}{L_2}\right)\frac{di_1}{dt} - \frac{M}{L_2} r_2 i_2 \tag{30}$$

If r_2 is small, equation (30) may be written

$$v_1 = r_1 i_1 + \left(L_1 - \frac{M^2}{L_2}\right)\frac{di_1}{dt}, \text{ approximately} \tag{31}$$

Equation (31) is of the same form as equation (1), page 129, for a simple series circuit containing resistance and self-inductance in series, except that $\left(L_1 - \frac{M^2}{L_2}\right)$ replaces L_1.

The effect of the short-circuited coupled circuit is to diminish the apparent self-inductance of the primary circuit. If the coefficient of coupling could be made unity, $\frac{M^2}{L_2}$ would be equal to L_1. In this case, the apparent inductance $\left(L_1 - \frac{M^2}{L_2}\right)$ of the primary circuit would be zero, and it would act like a circuit containing only pure resistance. Although it is not possible to make the apparent inductance of the primary circuit exactly zero, it may be made very low by interwinding the two circuits so as to make the magnetic leakage between them very small.

A closely coupled, low-resistance, short-circuited secondary winding may be used to suppress the arc when a relay circuit which carries direct current is opened. For this purpose the main relay winding would probably be wound on a copper cylinder which would serve as the low-resistance, closely coupled, secondary winding. This cylinder would have no effect on the operation of the relay so long as the exciting current was steady. When the exciting current varied, however, the current induced in the low-resistance, closely coupled winding would decrease the

apparent inductance of the exciting winding. The presence of the coupled winding would cause the exciting current to reach its final steady value more quickly after the circuit was closed By decreasing the apparent inductance of the exciting winding, it would also practically suppress the arc when the exciting circuit was broken. The effect of the coupled winding is to decrease the apparent time constant of the primary winding.

Leakage Inductance.—When dealing with certain circuits coupled by mutual inductance, it is often desirable to split the self-inductance of each winding into two parts. One part, called the *leakage inductance,* is produced by the leakage flux. The other part is due to that portion of the total flux of self-induction which links both windings. The treatment of the alternating-current power transformer and also of the induction motor is very much simplified by the use of this device.

Self-inductance L has been defined as the change in the flux linkages of a circuit per unit change in its current. The flux concerned includes *all* of the flux produced by the current when it acts alone. Similarly, the mutual inductance M of two circuits has been defined as the change produced in the flux linkages of one circuit by unit change of current in the other. In this case, the flux concerned includes only that portion of the total flux which *links both windings* when one alone carries current. It does not include any leakage flux, since the leakage flux, by its definition, cannot link the second circuit. Leakage inductance is defined in a similar manner. The leakage inductance S of a circuit is the rate of change of the leakage-flux linkages per unit current. The flux concerned in leakage inductance includes only that portion of the total flux which does not link the second circuit. Leakage inductance, as well as self-inductance and mutual inductance, is measured in henrys. The leakage inductance of a circuit is one henry when a change of one ampere in its current causes a change of 10^8 leakage-flux linkages. The leakage inductance of a circuit in henrys multiplied by $2\pi f$, where f is the frequency, is its leakage reactance in ohms. Just as $e_{1L} = L_1\frac{di_1}{dt}$ is the voltage drop produced in circuit 1 by self-inductance, and $e_{2M} = M\frac{di_1}{dt}$ is the voltage drop produced in

circuit 2 by mutual inductance, so $e_{1S} = S_1 \frac{di_1}{dt}$ is the voltage drop produced in circuit 1 by leakage inductance. When the inductances are constant and the currents are sinusoidal, the corresponding root-mean-square or effective values of the voltages are

$$
\begin{aligned}
E_{1L} &= 2\pi f L_1 I_1 = \omega L_1 I_1 \\
E_{2M} &= 2\pi f M I_1 = \omega M I_1 \\
E_{1S} &= 2\pi f S_1 I_1 = \omega S_1 I_1
\end{aligned}
$$

If the inductances are expressed in henrys, the currents in amperes and the frequency in cycles per second, the voltages are in volts.

Let φ_{1S} and φ_{2S} be the leakage fluxes for circuits 1 and 2. Assuming that all of the leakage flux links all of the turns of the circuit in which it is produced, the leakage inductances are

$$S_1 = N_1 \frac{d\varphi_{1S}}{di_1} 10^{-8} \text{ henrys} \tag{32}$$

$$S_2 = N_2 \frac{d\varphi_{2S}}{di_2} 10^{-8} \text{ henrys} \tag{33}$$

where the currents are expressed in amperes.

If the leakage fluxes per ampere are constant,

$$S_1 = N_1 \frac{\varphi_{1S}}{I_1} 10^{-8} \text{ henrys} \tag{34}$$

$$S_2 = N_2 \frac{\varphi_{2S}}{I_2} 10^{-8} \text{ henrys} \tag{35}$$

Relations among the Fluxes Corresponding to Self-inductance, Leakage Inductance and Mutual Inductance.—Consider two coupled circuits having N_1 and N_2 turns. Let the inductances be expressed in henrys. Assume the inductances to be constant. This is equivalent to saying that there is no magnetic material in the vicinity of the circuits.

For I_1 amperes in circuit 1, circuit 2 being open,

$\frac{L_1}{N_1} I_1 \times 10^8 = \varphi_1$ maxwells = total flux linking circuit 1 causing its self-inductance

$\frac{S_1}{N_1} I_1 \times 10^8 = \varphi_{1S}$ maxwells = leakage flux linking circuit causing its leakage inductance

$\frac{M}{N_2} I_1 \times 10^8 = \varphi_{1M}$ maxwells = flux linking circuit 2 causing the mutual inductance of circuit 1 on circuit 2

For I_2 amperes in circuit 2, circuit 1 being open,

$\frac{L_2}{N_2} I_2 \times 10^8 = \varphi_2$ maxwells = total flux linking circuit 2 causing its self-inductance

$\frac{S_2}{N_2} I_2 \times 10^8 = \varphi_{2S}$ maxwells = leakage flux linking circuit 2 causing its leakage inductance

$\frac{M}{N_1} I_2 \times 10^8 = \varphi_{2M}$ maxwells = flux linking circuit 1 causing the mutual inductance of circuit 2 on circuit 1

Obviously,

$\varphi_1 - \varphi_{1S} = \varphi_{1M}$ maxwells = part of φ_1 which also links circuit 2
$\varphi_2 - \varphi_{2S} = \varphi_{2M}$ maxwells = part of φ_2 which also links circuit 1

From the preceding equations it follows that

$$(L_1 - S_1) \frac{I_1}{N_1} \times 10^8 = \frac{M}{N_2} I_1 \times 10^8 = \varphi_{1M} \text{ maxwells}$$

$$L_1 - S_1 = \frac{N_1}{N_2} M \tag{36}$$

$$(L_2 - S_2) \frac{I_2}{N_2} \times 10^8 = \frac{M}{N_1} I_2 \times 10^8 = \varphi_{2M} \text{ maxwells}$$

$$L_2 - S_2 = \frac{N_2}{N_1} M \tag{37}$$

From equations (36) and (37),

$$(L_1 - S_1) \times (L_2 - S_2) = M^2 \tag{38}$$

and

$$M = \sqrt{(L_1 - S_1) \times (L_2 - S_2)} \tag{39}$$

but, since $L_1 - S_1 = L_1(1 - k_1)$ and $L_2 - S_2 = L_2(1 - k_2)$, equation (39) is equivalent to equation (17), page 191.

Now let circuits 1 and 2 carry $\bar{I}_1$ and $\bar{I}_2$ amperes, respectively, at the same time. In general, the currents are not in phase and the fluxes they produce are not in phase. They must be

added vectorially. Both the currents and fluxes must be expressed as vectors.

$$\bar{\varphi}_{1M} + \bar{\varphi}_{2M} = \left(\frac{M}{N_2}\bar{I}_1 + \frac{M}{N_1}\bar{I}_2\right)10^8 = \bar{\varphi}_{MR} \tag{40}$$

$$= \frac{M}{N_1 N_2}(\bar{I}_1 N_1 + \bar{I}_2 N_2)\,10^8 \text{ maxwells} \tag{41}$$

is the resultant mutual flux linking circuits 1 and 2 when they carry currents $\bar{I}_1$ and $\bar{I}_2$, respectively. $\bar{\varphi}_{MR}$ is the vector sum of the two component mutual fluxes $\bar{\varphi}_{1M}$ and $\bar{\varphi}_{2M}$ produced by the currents $\bar{I}_1$ and $\bar{I}_2$, respectively. $\bar{\varphi}_{MR}$ is the maximum or the root-mean-square value of the resultant mutual flux, according as the maximum or the root-mean-square values of the currents are used.

When both circuits carry current, it is seen that the resultant mutual flux is made up of two components, one, $\bar{\varphi}_{1M}$, produced by and in phase with $\bar{I}_1$, the other, $\bar{\varphi}_{2M}$, produced by and in phase with $\bar{I}_2$. The resultant ampere-turns acting on the magnetic circuit to produce the resultant mutual flux $\bar{\varphi}_{MR}$ are $(\bar{I}_1 N_1 + \bar{I}_2 N_2)$. [See equation (41).]

$$\bar{\varphi}_{MR} + \bar{\varphi}_{1S} = \bar{\varphi}_{1R} \tag{42}$$

is the resultant flux linking circuit 1 when both circuits carry current. It is the vector sum of the resultant mutual flux and the leakage flux of circuit 1. The total voltage induced in circuit 1 is produced by this flux. This voltage consists of two components, one due to the leakage flux $\bar{\varphi}_{1S}$, the other due to the resultant mutual flux $\bar{\varphi}_{MR}$.

$$\bar{\varphi}_{MR} + \bar{\varphi}_{2S} = \bar{\varphi}_{2R} \tag{43}$$

is the resultant flux linking circuit 2 when both circuits carry current. It is the vector sum of the resultant mutual flux and the leakage flux of circuit 2. The total voltage induced in circuit 2 is produced by this flux. This voltage consists of two components, one due to the leakage flux $\bar{\varphi}_{2S}$, the other due to the resultant mutual flux $\bar{\varphi}_{MR}$.

Voltages Induced in the Coils of a Two-winding Air-core Transformer.—Figure 56 shows a transformer with two coils wound on a core.

For the present purpose the core is assumed to be of non-magnetic material. When an iron core is used, the coefficients of self-induction and mutual induction are not constant. They vary with the saturation of the magnetic circuit. However, the presence of an iron core in commercial alternating-current power transformers does not introduce any serious difficulty, if the self-inductances of the primary and secondary windings are split up into two parts, one due to leakage flux and the other due to mutual flux. Sinusoidal current and voltage waves are assumed.

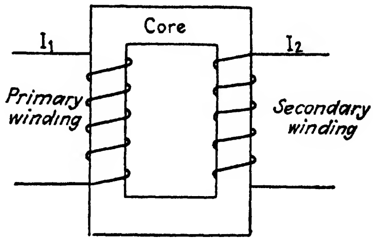


FIG. 56.

The total voltage drop in the primary winding due to self-inductance and mutual inductance, when both primary and secondary windings carry current, is

$$\bar{E}_1 = j\omega L_1\bar{I}_1 + j\omega M\bar{I}_2 \tag{44}$$

$$= j\omega\left(S_1\bar{I}_1 + M\bar{I}_1\frac{N_1}{N_2}\right) + j\omega M\bar{I}_2$$

$$= j\omega\, S_1\bar{I}_1 + j\frac{\omega M}{N_2}(\bar{I}_1N_1 + \bar{I}_2N_2) \tag{45}$$

Since $\bar{I}_1$ and $\bar{I}_2$ are not in phase, equations (44) and (45) and the equations which follow are true only when considered in a vector sense.

$M\bar{I}_1\frac{N_1}{N_2}$ is the mutual-flux linkages with circuit 1 due to $\bar{I}_1$.

$M\bar{I}_2$ is the mutual-flux linkages with circuit 1 due to $\bar{I}_2$.

$\frac{M}{N_2}(\bar{I}_1N_1 + \bar{I}_2N_2)$ is, therefore, the resultant mutual-flux linkages with circuit 1 due to the combined action of currents $\bar{I}_1$ and $\bar{I}_2$. Dividing this by N_1 gives the resultant mutual flux $\bar{\varphi}_{MR}$ due to the combined action of currents $\bar{I}_1$ and $\bar{I}_2$.

$$\bar{\varphi}_{MR} = \frac{M}{N_1N_2}(\bar{I}_1N_1 + \bar{I}_2N_2) \tag{46}$$

The first term, $j\omega S_1\bar{I}_1$, of the second member of equation (45) is the voltage drop produced in the primary winding by the

primary leakage flux, *i.e.*, by that portion of the total primary flux which does not link the secondary winding. The second term, $j\frac{\omega M}{N_2}(\bar{I}_1N_1 + \bar{I}_2N_2)$, is the voltage drop produced in the primary winding by the resultant mutual flux $\bar{\varphi}_{MR}$, due to the combined action of the primary and secondary currents $\bar{I}_1$ and $\bar{I}_2$.

The total voltage drop due to induction in the secondary winding is given by an expression similar to equation (45),

$$\bar{E}_2 = j\omega S_2\bar{I}_2 + j\frac{\omega M}{N_1}(\bar{I}_2N_2 + \bar{I}_1N_1) \qquad (47)$$

Since power is absorbed by the primary winding and is given out by the secondary winding, the primary and secondary currents $\bar{I}_1$ and $\bar{I}_2$ must be, in a general sense, in phase opposition and must, therefore, produce component fluxes which are also in a general sense in phase opposition. The current $\bar{I}_1$ may be considered to be made up of two components, one of which is equal to $\bar{I}_2\frac{N_2}{N_1}$ and in phase opposition to $\bar{I}_2$. Call this component $\bar{I}_1'$. Call the other component $\bar{I}_\varphi$. Then,

$$\bar{I}_1'N_1 = -\bar{I}_2N_2 \qquad (48)$$

and

$$\bar{E}_1 = j\omega S_1\bar{I}_1 + j\frac{\omega M}{N_2}\{N_1(\bar{I}_1' + \bar{I}_\varphi) + N_2\bar{I}_2\} \qquad (49)$$

$$= j\omega S_1\bar{I}_1 + j\frac{\omega M}{N_2}N_1\bar{I}_\varphi \qquad (50)$$

The component $\bar{I}_\varphi$ of the primary current is called the magnetizing current. The resultant mutual flux $\bar{\varphi}_{MR}$ may be considered due to $\bar{I}_\varphi$, since the effect of the other component, $\bar{I}_1'$, of the primary current is balanced, so far as the production of flux is concerned, by the secondary current $\bar{I}_2$. $\bar{I}_1'$ and $\bar{I}_2$ produce equal and opposite ampere turns, equation (48), and therefore cannot cause any resultant mutual flux.

In an air-core transformer, the so-called magnetizing component $\bar{I}_\varphi$ of the primary current is very large, owing to the low permeability of the magnetic circuit. If, however, the frequency is very high, the flux required for a fixed primary voltage is much reduced. Under this condition, the magnetizing component

of the primary current may be reduced to a reasonable value. For an iron-core transformer of good design, the component current $\bar{I}_\varphi$ is very small at commercial frequencies, such as are used for power generation and transmission. If the primary resistance and primary leakage reactance of such a transformer are made small, $\bar{I}_\varphi$ is nearly constant and independent of the load, varying only one or two per cent from no load to full load.

Vector Diagrams of a Two-winding Air-core Transformer.—The air-core transformer is an important example of two inductively coupled circuits, each having resistance and self-inductance. The significance of the components in the equations for the voltages induced in the windings of circuits having resistance, self-inductance and mutual inductance will be made clearer by a study of the vector diagram of an air-core transformer.

Consider an air-core transformer with a load on its secondary. The impressed voltage drop across the primary is equal to the primary resistance drop plus the voltage drops in the primary winding caused by its self-inductance and by the mutual inductance effect of the secondary. The voltage drop across the secondary is equal to the secondary resistance drop plus the voltage drops in the secondary winding caused by its self-inductance and the mutual inductance effect of the primary. All equations must be considered in a vector sense.

The voltage drop impressed across the primary winding of the transformer is

$$\bar{V}_1 = r_1\bar{I}_1 + j\omega L_1\bar{I}_1 + j\omega M\bar{I}_2 \tag{51}$$

$$\bar{E}_{1R} = j\omega L_1\bar{I}_1 + j\omega M\bar{I}_2 \tag{52}$$

$$\bar{\varphi}_{1R} = \bar{\varphi}_{1L} + \bar{\varphi}_{2M} = \frac{L_1\bar{I}_1}{N_1} + \frac{M\bar{I}_2}{N_1} \tag{53}$$

Equation (52) is an equation of induced voltage drops. Equation (53) is an equation of the fluxes producing the voltages given in equation (52). The corresponding voltages and fluxes are placed directly under each other in these two equations.

The vector diagram corresponding to equation (51) is shown in Fig. 57. The primary and secondary currents are assumed to be 145 degrees out of phase, with the primary current leading. The number of turns in the primary winding is assumed equal to the number in the secondary winding.

If the primary self-induced voltage drop $j\omega L_1\bar{I}_1$ is split up into two parts, one, $j\omega S_1\bar{I}_1$, due to leakage flux, the other, $j\omega M\bar{I}_1\dfrac{N_1}{N_2}$, due to the component $\bar{\varphi}_{1M}$ of the resultant mutual

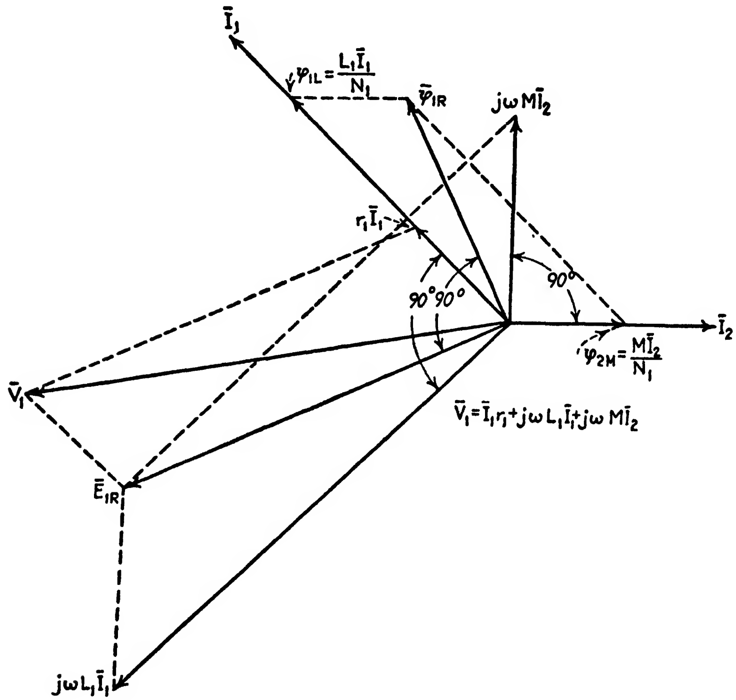


FIG. 57.

flux which is produced by the primary current, equation (51) becomes

$$\bar{V}_1 = r_1\bar{I}_1 + j\omega\left(S_1\bar{I}_1 + M\bar{I}_1\frac{N_1}{N_2}\right) + j\omega M\bar{I}_2 \tag{54}$$

The total induced primary voltage drop $\bar{E}_{1R}$, *i.e.*, the voltage drop produced in the primary winding by the total resultant primary flux, is given in equation (55). Equation (56) shows the component fluxes corresponding to the component voltage drops given in equation (55). The component fluxes, in equation (56), are placed directly below the corresponding component voltage drops in equation (55).

$$\bar{E}_{1R} = j\omega S_1\bar{I}_1 + j\omega M\bar{I}_1\frac{N_1}{N_2} + j\omega M\bar{I}_2 \tag{55}$$

$$\bar{\varphi}_{1R} = \bar{\varphi}_{1S} + \bar{\varphi}_{1M} + \bar{\varphi}_{2M} \tag{56}$$

$$= \bar{\varphi}_{1S} + \bar{\varphi}_{MR} \tag{57}$$

The total primary induced voltage drop $\bar{E}_{1R}$ may be considered to be made up of two components, *viz.*, the voltage drop produced by the primary leakage flux and a voltage drop produced by a flux $\bar{\varphi}_{MR}$, usually called the mutual flux, which is the resultant of the two component mutual fluxes $\bar{\varphi}_{1M}$ and

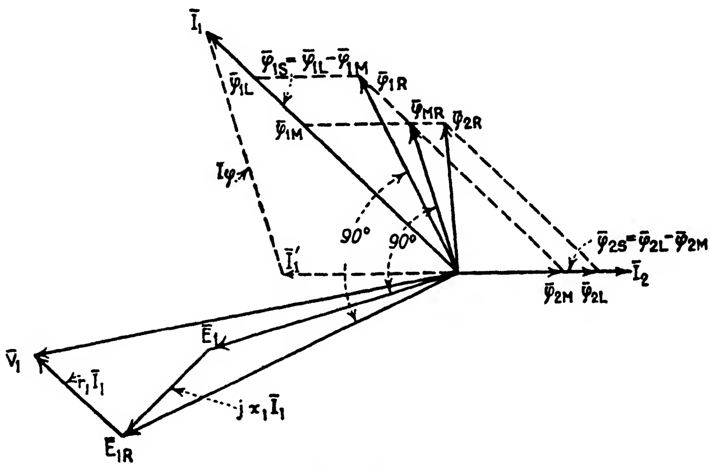


Fig. 58.

$\bar{\varphi}_{2M}$, produced by the primary and secondary currents, respectively. The voltage corresponding to the flux $\bar{\varphi}_{MR}$ is marked $\bar{E}_1$ on Fig. 58. This voltage is usually called the primary induced voltage drop. In reality, it is only a part of the primary induced voltage drop. The other part is due to the leakage flux and is replaced by a primary leakage reactance drop, $jx_1\bar{I}_1 = j\omega S_1\bar{I}_1$, on the transformer diagram as ordinarily drawn. The actual primary induced voltage drop is $\bar{E}_{1R}$. Equations similar to equations (55), (56) and (57), with the subscripts 1 and 2 interchanged, hold for the secondary.

Figure 58 shows the component fluxes corresponding to the terms of equations (55) and (56). The secondary leakage flux $\bar{\varphi}_{2S}$ and the flux $\bar{\varphi}_{2L}$, corresponding to the self-induced voltage in

the secondary, are added. $\bar{\varphi}_{1R} = \bar{\varphi}_{MR} + \bar{\varphi}_{1S}$ and $\bar{\varphi}_{2R} = \bar{\varphi}_{MR} + \bar{\varphi}_{2S}$ are the total resultant fluxes linking the primary and secondary windings, respectively. [See equations (42) and (43), page 198.]

As was stated on page 200, the primary current may be resolved into two components: one, $\bar{I}_1'$, which is opposite to the secondary current $\bar{I}_2$ and just balances the demagnetizing effect of that current in producing mutual flux, the other, $\bar{I}_\varphi$, which may be considered to produce the resultant mutual flux $\bar{\varphi}_{MR}$. In an air-core transformer, $\bar{I}_\varphi$ and $\bar{E}_1$ are in quadrature, since there

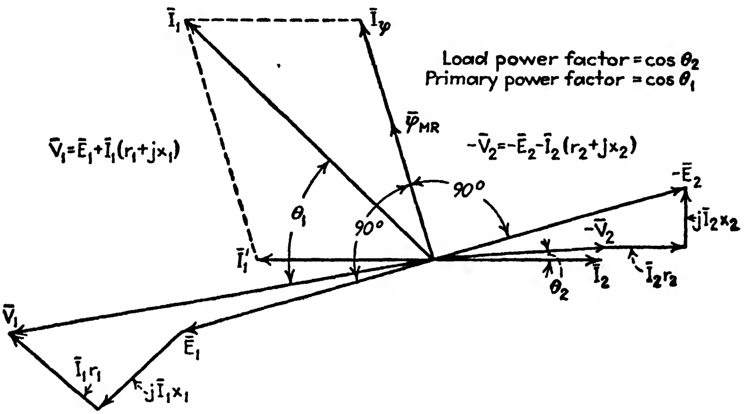


FIG. 59.

is no core loss. The only losses are the copper losses. These are taken care of by the $\bar{I}_1r_1$ and $\bar{I}_2r_2$ drops in the primary and secondary windings. In Fig. 58, $\bar{I}_1' = -\bar{I}_2$, since the ratio of the turns in the primary and secondary windings was assumed to be unity, *i.e.*, N_1 was assumed to be equal to N_2. Since $\bar{I}_\varphi$ produces the resultant mutual flux $\bar{\varphi}_{MR}$, it must be in phase with this flux.

The ordinary vector diagram of an air-core transformer is shown in Fig. 59. This diagram is derived directly from Fig. 58 by dividing the primary current into the two components $\bar{I}_1'$ and $\bar{I}_\varphi$ and replacing the voltage drops due to the leakage inductances S_1 and S_2 by the leakage reactance drops $j\bar{I}_1x_1$ and $j\bar{I}_2x_2$. $\bar{E}_1$ is the voltage drop produced in the primary winding by the resultant mutual flux φ_{MR}. The corresponding voltage drop in the secondary winding is $\bar{E}_2$. $-\bar{E}_2$, that is, the voltage rise, is used on Fig. 59 in place of the voltage drop to keep the

diagram more open. Also, since power is absorbed on the primary side of a transformer and delivered on the secondary side, it is better to draw $\bar{E}_1$ as a voltage drop and $-\bar{E}_2$ as a voltage rise. Since $\bar{E}_1$ and $\bar{E}_2$ are produced by the same flux, *i.e.*, by the resultant mutual flux φ_{MR}, they must be in phase and their magnitudes must be in the same ratio as the number of primary and secondary turns. $\bar{E}_1$ and $-\bar{E}_2$ are therefore in opposite phase.

$$\frac{E_1}{E_2} = \frac{N_1}{N_2} = a \tag{58}$$

is known as the *ratio of transformation of the transformer.*

In Fig. 59, $\bar{E}_1$ is equal in magnitude to $-\bar{E}_2$, since N_1 and N_2 are assumed equal. Since power is delivered on the secondary side, drawing $-\bar{E}_2$ as a voltage rise puts its energy component in phase with $\bar{I}_2$. Drawing $\bar{E}_1$ as a voltage drop puts its active component in phase with $\bar{I}_1$. The secondary terminal voltage rise is equal to the voltage rise $-\bar{E}_2$ minus the secondary leakage reactance and resistance drops.

Example of the Determination of the Self-inductance, Leakage Inductance and Mutual Inductance of a Two-winding Air-core Transformer.—When the secondary winding of a certain air-core transformer is on open circuit and 50 volts at 60 cycles are impressed on its primary winding, it takes 200 watts at 0.7 power factor. The voltage across the secondary winding under these conditions, measured by a voltmeter taking negligible current, is 64.3 volts. The primary winding has 2000 turns and the ratio of the number of turns in the primary and secondary windings is $\frac{N_1}{N_2} = 0.5$. What are the self-inductance and leakage inductance of the primary winding in henrys? What is the mutual inductance of the two windings in henrys? What is the root-mean-square or effective voltage induced in the primary winding by the total primary flux? What is the maximum value of this flux?

$$\begin{aligned} V_1 I_1 \cos\theta_1 &= P_1 \\ 50 \times I_1 \times 0.7 &= 200 \\ I_1 &= \frac{200}{50 \times 0.7} \\ &= 5.71 \text{ amperes} \end{aligned}$$

Since there is no iron core, the entire power taken by the transformer, when there is no load on the secondary, is primary copper loss.

$$r_1 = \frac{P_1}{I_1^2} = \frac{200}{(5.71)^2} = 6.13 \text{ ohms}$$

Taking I_1 as the axis of reals,

$$\begin{aligned}
\bar{I}_1 &= 5.71(1 + j0) \\
\bar{V}_1 &= 50(\cos\theta_1 + j\sin\theta_1) \\
&= 50(0.7 + j0.7141) \\
&= 35 + j35.71
\end{aligned}$$

Since there is no power consumed, except that due to copper loss in the primary winding, the reactive component of the voltage drop across the primary winding must be entirely due to primary self-inductance.

$$\begin{aligned}
E_1 &= \omega L_1 I_1 = 35.71 \text{ volts} \\
L_1 &= \frac{35.71}{377 \times 5.71} = 0.0166 \text{ henry} \\
E_2 &= \omega M I_1 \\
M &= \frac{64.3}{377 \times 5.71} = 0.0299 \text{ henry} \\
L_1 - S_1 &= M\frac{N_1}{N_2} \\
S_1 &= L_1 - M\frac{N_1}{N_2} \\
&= 0.0166 - 0.0299 \times 0.5 \\
&= 0.0017 \text{ henry} \\
E_1 &= 4.44 N_1 \varphi_m f 10^{-8} \text{ (See page 49)} \\
\varphi_m &= \frac{E_1}{4.44 N_1 f 10^{-8}} \\
&= \frac{35.71 \times 10^8}{4.44 \times 2000 \times 60} \\
&= 6.70 \times 10^3 \text{ maxwells}
\end{aligned}$$

Two Circuits Coupled by Mutual Inductance Equivalent to T-connected Circuit.—When the coefficients of self-inductance and mutual inductance are constant, the voltage drops across the terminals of the two circuits may be written in vector form.

This assumes sinusoidal currents and voltages. [See equations (25) and (26).]

$$\bar{V}_1 = r_1\bar{I}_1 + j\omega L_1\bar{I}_1 + j\omega M\bar{I}_2 \tag{59}$$
$$\bar{V}_2 = r_2\bar{I}_2 + j\omega L_2\bar{I}_2 + j\omega M\bar{I}_1 \tag{60}$$

These are the equations for an air-core transformer with two windings. By a simple transformation, equations (59) and (60) may be reduced to those of a T-connected circuit with simple inductive coupling. If $j\omega M\bar{I}_1$ is added to and subtracted from the right-hand member of equation (59), it becomes

$$\bar{V}_1 = r_1\bar{I}_1 + j\omega(L_1 - M)\bar{I}_1 + j\omega M(\bar{I}_1 + \bar{I}_2) \tag{61}$$

By a similar transformation, equation (60) becomes

$$\bar{V}_2 = r_2\bar{I}_2 + j\omega(L_2 - M)\bar{I}_2 + j\omega M(\bar{I}_1 + \bar{I}_2) \tag{62}$$

Equations (61) and (62) are the equations for the input and output voltage drops, respectively, of the inductively coupled circuit shown in Fig. 60.

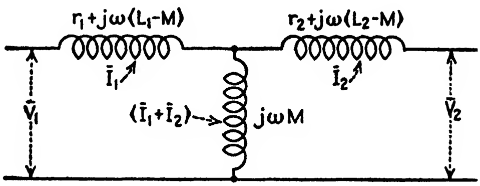


FIG. 60.

If M is larger than L_1, then $(L_1 - M)$ is negative. Under this condition, it is equivalent to a capacitance. It differs from a real capacitance in that it varies directly as the frequency, instead of inversely as the frequency as in the case of real capacitance. If M is smaller than L_1, $(L_1 - M)$ is positive and is equivalent to real inductance. Similar statements hold in regard to $(L_2 - M)$.

The transformation just explained is often convenient and with a slight modification is frequently used in handling the two-circuit power transformer.

Transformer with Ratio of Transformation of Unity and Ideal Transformer Replace Two-circuit Transformer.—An equivalent transformer has the same losses and the same coupling as the

one it replaces. An ideal transformer is one which has neither losses nor magnetic leakage. It is often convenient to replace a given two-circuit transformer by an equivalent transformer with a ratio of transformation of unity, *i.e.*, with a ratio of turns on the input and output sides equal to unity, and by an ideal transformer with a ratio of transformation equal to the ratio of the original transformer.

If the secondary circuit of a transformer is replaced by an equivalent one with the same number of turns as the primary, all secondary voltages must be multiplied by $\frac{N_1}{N_2}$ and the secondary current must be multiplied by $\frac{N_2}{N_1}$. When these changes are made, the equations of the equivalent air-core transformer with a ratio of transformation of unity are, from equations (59) and (60), page 207,

$$\bar{V}_1 = r_1\bar{I}_1 + j\omega L_1\bar{I}_1 + j\omega\frac{N_1}{N_2}M\left(\bar{I}_2\frac{N_2}{N_1}\right) \tag{63}$$

$$\bar{V}_2\frac{N_1}{N_2} = \frac{N_1}{N_2}(r_2\bar{I}_2 + j\omega L_2\bar{I}_2 + j\omega M\bar{I}_1)$$
$$= \left(\frac{N_1}{N_2}\right)^2 r_2\left(\frac{N_2}{N_1}\bar{I}_2\right) + j\omega\left(\frac{N_1}{N_2}\right)^2 L_2\left(\frac{N_2}{N_1}\bar{I}_2\right) + j\omega\left(\frac{N_1}{N_2}M\right)\bar{I}_1 \tag{64}$$

$\left(\frac{N_1}{N_2}\right)^2 r_2$ and $\left(\frac{N_1}{N_2}\right)^2 L_2$ are, respectively, the resistance and the self-inductance of the equivalent secondary and $\frac{N_2}{N_1}\bar{I}_2$ is the equivalent secondary current.

Add to and subtract from equation (63) $j\omega\left(\frac{N_1}{N_2}M\right)\bar{I}_1$, and add to and subtract from equation (64) $j\omega\left(\frac{N_1}{N_2}M\right)\left(\frac{N_2}{N_1}\bar{I}_2\right)$.

$$\bar{V}_1 = r_1\bar{I}_1 + j\omega\left(L_1 - M\frac{N_1}{N_2}\right)\bar{I}_1 + j\omega\left(\frac{N_1}{N_2}M\right)\left(\bar{I}_1 + \frac{N_2}{N_1}\bar{I}_2\right) \tag{65}$$

$$\left(\frac{N_1}{N_2}\right)\bar{V}_2 = \left(\frac{N_1}{N_2}\right)^2 r_2\left(\frac{N_2}{N_1}\bar{I}_2\right) + j\omega\left[\left(\frac{N_1}{N_2}\right)^2 L_2 - \frac{N_1}{N_2}M\right]\left(\frac{N_2}{N_1}\bar{I}_2\right)$$
$$+ j\omega\left(\frac{N_1}{N_2}M\right)\left[\bar{I}_1 + \left(\frac{N_2}{N_1}\bar{I}_2\right)\right] \tag{66}$$

The diagram of the equivalent T circuit of the transformer with the ratio of transformation of unity and its ideal transformer is shown in Fig. 61. In this figure, for simplicity, $\frac{N_1}{N_2}$ is replaced by a.

A is the equivalent T-connected circuit for the transformer with a ratio of transformation of unity, *i.e.*, with $N_1 = N_2$. B is the ideal transformer with neither losses nor leakage and with a ratio of transformation equal to that of the actual transformer.

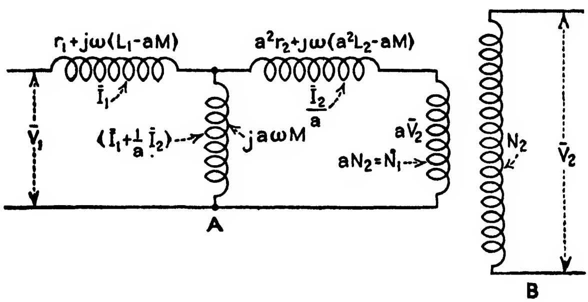


FIG. 61.

Although L_1, L_2 and M are not constant in an iron-core transformer, the leakage fluxes corresponding to the leakage inductances $(L_1 - aM)$ and $(a^2L_2 - aM)$ lie largely in air. For this reason, the leakage inductances are sensibly constant, although the saturation of the core varies. By the addition of a conductance in parallel with $ja\omega M$ to take care of the core loss in the iron core, Fig. 61 may be made to represent a two-circuit, iron-core transformer.

Uses of Transformers.—Transformers are used for two widely different purposes. For power transmission, the transformer is used to step up the generated voltage in order that the transmission may be efficiently accomplished with small current and a small amount of copper. In connection with communication circuits, transformers are used to adjust the impedances of the input and output circuits in order to attain the condition of maximum power transfer between the transmitting and receiving circuits. This occurs when the impedances of the input and output circuits are conjugates. Under this condition, the

efficiency would be fifty per cent, if it were not for the iron losses in the iron core which is used. These losses make the efficiency somewhat less than fifty per cent under the condition of maximum power transfer between the two circuits. For the purpose of power transmission, the impedances are not adjusted for the condition of maximum power transfer between the input and output circuits and, indeed, the impedances could not be so adjusted on account of the varying impedance of the power load. The thing that is desired in a power transformer is maximum efficiency at reasonable cost.

The input impedance of an air-core transformer may be found from equations (59) and (60), page 207, by replacing V_2 by $-\bar{I}_2\bar{Z}_L$, where $\bar{Z}_L$ is the load impedance, and then dividing $\bar{V}_1$ by $\bar{I}_1$.

Putting $\bar{V}_2$ in equation (60) equal to $-\bar{I}_2\bar{Z}_L$, and solving for $\bar{I}_2$, gives

$$\bar{I}_2 = \frac{-j\omega M\bar{I}_1}{r_2 + \bar{Z}_L + j\omega L_2}$$

Substituting this value of $\bar{I}_2$ in equation (59), and solving for $\frac{\bar{V}_1}{\bar{I}_1}$, gives

$$\frac{\bar{V}_1}{\bar{I}_1} = r_1 + j\omega L_1 + \frac{\omega^2 M^2}{r_2 + \bar{Z}_L + j\omega L_2}$$
$$= \frac{r_1(r_2 + \bar{Z}_L + j\omega L_2) + j\omega L_1 r_2 + j\omega L_1\bar{Z}_L - \omega^2 L_1 L_2 + \omega^2 M^2}{r_2 + \bar{Z}_L + j\omega L_2} \quad (67)$$

If r_1 and r_2 are small,

$$\frac{\bar{V}_1}{\bar{I}_1} = \frac{j\omega L_1\bar{Z}_L - \omega^2 L_1 L_2 + \omega^2 M^2}{\bar{Z}_L + j\omega L_2}, \text{ approximately} \quad (68)$$

If the magnetic leakage is negligible, $L_1L_2 = M^2$ [see equation (17), page 191] and

$$\frac{\bar{I}_1}{\bar{V}_1} = \frac{\bar{Z}_L + j\omega L_2}{j\omega L_1\bar{Z}_L} = \frac{1}{j\omega L_1} + \frac{L_2}{L_1}\frac{1}{\bar{Z}_L} \quad (69)$$

If L_1 and L_2 are large, $\frac{1}{j\omega L_1}$ is negligible compared with $\frac{L_2}{L_1}\frac{1}{\bar{Z}_L}$, especially when ω is large, as it would be in the case of a communication transformer. In this case,

$$\frac{\bar{V}_1}{\bar{I}_1} = \frac{L_1}{L_2}\bar{Z}_L = \left(\frac{N_1}{N_2}\right)^2 \bar{Z}_L \tag{70}$$

If the losses and leakage inductances of the transformer are negligible, the apparent impedance of the load, considered from the primary side of the transformer, is equal to the actual load impedance multiplied by the square of the ratio of the numbers of primary and secondary turns in the windings of the transformer. By the use of a transformer with a suitable ratio of primary and secondary turns, it is possible to make the apparent impedance of the receiver of a communication circuit approximately equal to the impedance of the transmitter and thus to realize approximately the condition of maximum power transmission. In practice, exact equality cannot be attained because the losses and magnetic leakage of a transformer cannot be made zero.

CHAPTER VII

KIRCHHOFF'S LAWS, IMPEDANCES IN SERIES AND PARALLEL, EFFECTIVE RESISTANCE AND REACTANCE

Kirchhoff's Laws.—Kirchhoff's laws may be applied to single-phase or to polyphase alternating-current circuits as well as to direct-current circuits. When applied to alternating-current circuits, either instantaneous values of currents and voltages must be used in the equations, or all currents and voltages must be considered in a vector sense and must be referred to some conveniently chosen axis of reference. When instantaneous values are used, the resulting equations are algebraic equations. When vectors are used, the resulting equations are vector equations, and all currents, voltages and impedances must be expressed in their complex form and all referred to the same axis of reference. The solution of the resulting equations gives the unknown quantities in their complex form.

Kirchhoff's laws applied to an alternating-current circuit may be stated as follows:

For instantaneous values:

(*a*) The algebraic sum of the instantaneous values of all currents flowing toward any junction point in a circuit is zero at every instant.

$$i_1 + i_2 + i_3 + \cdots + i_n = 0$$
$$\Sigma_1^n\, i = 0 \qquad (1)$$

(*b*) The total rise or fall of potential at any instant in going around any closed circuit is zero.

$$e_1 + e_2 + e_3 + \cdots + e_n = 0$$
$$\Sigma_1^n\, e_{rise} = \Sigma_1^n\, e_{fall} = 0 \qquad (2)$$

For vectors:

(*a*) The vector sum of all currents considered toward any junction point in a circuit is zero.

$$\bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \cdots + \bar{I}_n = 0$$
$$\Sigma_1^n \bar{I} = 0 \tag{3}$$

(*b*) The vector sum of all the potential rises or potential falls taken in the same direction around any closed circuit is zero.

$$\bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \cdots + \bar{E}_n = 0$$
$$\Sigma_1^n \bar{E}_{rise} = \Sigma_1^n \bar{E}_{fall} = 0 \tag{4}$$

Kirchhoff's laws hold under all conditions for instantaneous values, but when applied to vectors, sinusoidal currents and voltages are presupposed. The current and voltage of a circuit can both be simple harmonic functions of time only when the resistances, inductances and capacitances of the circuit are constant and independent of current strength. When the resistances, inductances and capacitances of a circuit are constant, Kirchhoff's laws may be applied to it, when the currents and voltages are not simple harmonic functions of time, by resolving the currents and voltages into their Fourier series and then considering the fundamentals and harmonics separately. The resultant current or voltage in any branch may then be found in the usual way by taking the square root of the sum of the squares of the root-mean-square values of the fundamental and harmonics.

When vectors are used to represent equivalent sine waves, the application of Kirchhoff's laws to a circuit may and usually does give inaccurate and generally useless results, especially when there is capacitance as well as inductance in the circuit.

The solution of series and parallel circuits and networks involves Kirchhoff's laws. The laws have already been used in Chapter VI. They will be used again in Chapter X in the solution of polyphase circuits.

Impedances in Series.—The current in a series circuit is the same in all parts and the resultant voltage drop across the entire circuit is equal to the vector sum of the voltage drops in its component parts. This may be expressed analytically by the following equations:

$$\bar{I}_0 = \bar{I}_1 = \bar{I}_2 = \bar{I}_3 = \text{etc.} \tag{5}$$
$$\bar{V}_0 = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \text{etc.} \tag{6}$$
$$= \bar{I}_0\bar{z}_1 + \bar{I}_0\bar{z}_2 + \bar{I}_0\bar{z}_3 + \text{etc.} \tag{7}$$

where $\bar{I}_0$ and $\bar{V}_0$ are the resultant current and the resultant voltage drop across the entire circuit. $\bar{I}$, $\bar{E}$ and $\bar{z}$ with the subscripts 1, 2, 3 etc. are the currents, voltage drops and impedances for the component parts of the circuit. The summations in equations (6) and (7) must be made in a vector sense.

It is common practice to use a dot or a short line over a letter, when the latter is used to represent a vector or a complex quantity, in all cases where confusion or misunderstanding would result without such designation. In all alternating-current

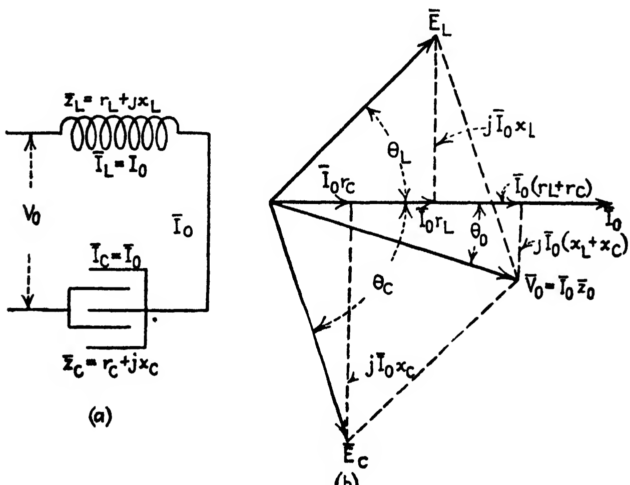


FIG. 62.

work, currents and voltages must be added or subtracted vectorially. No special designation, therefore, is necessary in most cases to indicate this. However, for the sake of greater clearness, a short line is used over all letters which represent either vectors or complex quantities. Power is not a vector, neither is resistance nor reactance, but resistance and reactance drops are vectors. Impedance is a complex quantity. Impedances, therefore, must be handled like vectors so far as operations of addition, subtraction, multiplication and division are concerned.

Consider a circuit consisting of an inductive impedance z_L in series with a capacitive impedance z_C. The diagram of connections is shown in Fig. 62(a). The inductive part of the circuit has a resistance r_L and an inductive reactance $x_L = 2\pi fL \doteq \omega L$. The

capacitive part of the circuit has a resistance r_C and a capacitive reactance $x_C = \dfrac{-1}{2\pi fC} = \dfrac{-1}{\omega C}$. The vector diagram of the circuit is shown in Fig. 62(*b*). *On Fig. 62 and in the following equations, it must be remembered that x_C is actually negative as defined.*

Referring to Fig. 62(*b*), $E_L = I_0\sqrt{r_L^2 + x_L^2}$ is the voltage drop across the inductive impedance z_L. This drop is made up of two parts, one, $I_0 r_L$, in phase with the current, the other, $I_0 x_L$, in quadrature with the current and *leading* it by 90 degrees. The voltage drop E_L leads the current by an angle whose tangent is $\dfrac{x_L}{r_L} = \dfrac{\omega L}{r_L}$. $E_C = I_0\sqrt{r_C^2 + x_C^2}$ is the voltage drop across the capacitive impedance z_C. This drop, like the voltage drop E_L across the inductive impedance z_L, is made up of two parts: one, $I_0 r_C$, in phase with the current, the other, $I_0 x_C = \dfrac{-I_0}{\omega C}$, in quadrature with the current but *lagging* it by 90 degrees. The voltage drop E_C leads the current by an angle θ_C whose tangent is

$$\frac{x_C}{r_C} = \frac{-1}{\omega r_C C}$$

Since θ_C is negative, E_C actually lags the current.

The resultant voltage drop $\bar{V}_0$ across the two impedances in series is equal to the vector sum of the voltage drops $\bar{E}_L$ and $\bar{E}_C$. This resultant voltage drop is made up of two parts, $\bar{I}_0 r_0 = \bar{I}_0(r_L + r_C)$ and $\bar{I}_0 x_0 = \bar{I}_0(x_L + x_C)$, respectively in phase and in quadrature with the current. Whether the resultant quadrature component leads or lags the current depends on the relative magnitudes of the inductive and capacitive reactances, x_L and x_C. If x_L is larger than x_C, it leads. On the other hand, if x_C is larger than x_L, it lags.

From Fig. 62(*b*) it is obvious that

$$\begin{aligned} V_0 \cos \theta_0 &= I_0 r_0 \\ &= I_0(r_L + r_C) \qquad (8) \\ V_0 \sin \theta_0 &= I_0 x_0 \\ &= I_0(x_L + x_C) \qquad (9) \end{aligned}$$

In general, if there are k impedances in series, it is clear from equations (8) and (9) that

$$V_0 \cos \theta_0 = I_0 r_0 = I_0(r_1 + r_2 + \cdots + r_k) \tag{10}$$
$$V_0 \sin \theta_0 = I_0 x_0 = I_0(x_1 + x_2 + \cdots + x_k) \tag{11}$$

and

$$r_0 = r_1 + r_2 + \cdots + r_k = \Sigma_1^k r \tag{12}$$
$$x_0 = x_1 + x_2 + \cdots + x_k = \Sigma_1^k x \tag{13}$$

For a series circuit, therefore, the resultant resistance is equal to the sum of the resistances of its separate parts. Similarly, the resultant reactance of a series circuit is equal to the sum of the reactances of its separate parts. A series circuit, therefore, acts like a simple circuit having a single resistance r_0 in series with a single reactance x_0. Resistances are always positive. Reactances, on the other hand, may be either positive or negative according as they are due to inductance or to capacitance. It is obvious, therefore, that reactances must be added algebraically, *i.e.*, with regard to their signs.

Since $V_0 \cos \theta_0 = I_0 r_0$ and $V_0 \sin \theta_0 = I_0 x_0$ are two quadrature components of the voltage V_0, it follows that

$$V_0 = I_0\sqrt{r_0^2 + x_0^2} = I_0 z_0 \tag{14}$$

and

$$I_0 = \frac{V_0}{\sqrt{r_0^2 + x_0^2}} = \frac{V_0}{z_0} \tag{15}$$

The current in amperes is always given by the voltage in volts impressed on the circuit divided by the resultant impedance in ohms. The resultant impedance z_0 is not the sum of the separate impedances except in complex.

$$\begin{aligned} z_0 &= \sqrt{(r_1 + r_2 + \cdots + r_k)^2 + (x_1 + x_2 + \cdots + x_k)^2} \\ &= \sqrt{(\Sigma_1^k r)^2 + (\Sigma_1^k x)^2} \end{aligned} \tag{16}$$

Referring to Fig. 62(*b*), it is evident that

$$\tan \theta_0 = \frac{x_0}{r_0} \tag{17}$$

$$\cos \theta_0 = \frac{r_0}{z_0} \tag{18}$$

$$\sin \theta_0 = \frac{x_0}{z_0} \tag{19}$$

Complex Method.—Since the current is the same in all parts of a series circuit, *i.e.*, it is common to all parts of the circuit,

it is taken as the axis of reference, *i.e.*, as the axis of reals. In complex, the voltage drops $\bar{E}_L$ and $\bar{E}_C$ across the two parts of the series circuit shown in Fig. 62 are

$$\bar{E}_L = \bar{I}_0\bar{z}_L = \bar{I}_0(r_L + jx_L) \tag{20}$$

$$\bar{E}_C = \bar{I}_0\bar{z}_C = \bar{I}_0(r_C + jx_C) \tag{21}$$

$$\begin{aligned}\bar{V}_0 = \bar{E}_L + \bar{E}_C &= \bar{I}_0\{(r_L + r_C) + j(x_L + x_C)\}\\ &= \bar{I}_0(r_0 + jx_0) = \bar{I}_0\bar{z}_0\end{aligned} \tag{22}$$

$$z_0 = r_0 + jx_0 \tag{23}$$

$$\tan\theta_0 = \frac{x_L + x_C}{r_L + r_C} = \frac{x_0}{r_0} \tag{24}$$

$$\cos\theta_0 = \frac{r_L + r_C}{\sqrt{(r_L + r_C)^2 + (x_L + x_C)^2}} = \frac{r_0}{z_0} \tag{25}$$

$$\sin\theta_0 = \frac{x_L + x_C}{\sqrt{(r_L + r_C)^2 + (x_L + x_C)^2}} = \frac{x_0}{z_0} \tag{26}$$

In general, if there are k impedances in series,

$$\begin{aligned}\bar{E}_1 &= \bar{I}_0\bar{z}_1 = \bar{I}_0(r_1 + jx_1)\\ \bar{E}_2 &= \bar{I}_0z_2 = \bar{I}_0(r_2 + jx_2)\\ &\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\\ \bar{E}_k &= \bar{I}_0\bar{z}_k = \bar{I}_0(r_k + jx_k)\\ \bar{E}_1 + \bar{E}_2 + \cdots + \bar{E}_k &= \bar{I}_0\bar{z}_1 + \bar{I}_0\bar{z}_2 + \cdots + \bar{I}_0\bar{z}_k\\ \bar{V}_0 = \Sigma_1^k\bar{E} &= \bar{I}_0\Sigma_1^k\bar{z}\\ &= \bar{I}_0\bar{z}_0\\ &= \bar{I}_0\{(r_1 + r_2 + \cdots + r_k)\\ &\qquad + j(x_1 + x_2 + \cdots + x_k)\}\\ &= \bar{I}_0(\Sigma_1^k r + j\Sigma_1^k x) \qquad (27)\\ &= \bar{I}_0(r_0 + jx_0) \qquad (28)\end{aligned}$$

The two terms $\bar{I}_0r_0$ and $j\bar{I}_0x_0$, in equation (28), are, respectively, the active and the reactive components of the resultant voltage drop across the circuit.

$$\tan\theta_0 = \frac{x_0}{r_0} = \frac{\Sigma_1^k x}{\Sigma_1^k r} \tag{29}$$

$$\cos\theta_0 = \frac{r_0}{\sqrt{r_0^2 + x_0^2}} = \frac{r_0}{z_0} = \frac{\Sigma_1^k r}{\sqrt{(\Sigma_1^k r)^2 + (\Sigma_1^k x)^2}} \tag{30}$$

$$\sin\theta_0 = \frac{x_0}{\sqrt{r_0^2 + x_0^2}} = \frac{x_0}{z_0} = \frac{\Sigma_1^k x}{\sqrt{(\Sigma_1^k r)^2 + (\Sigma_1^k x)^2}} \tag{31}$$

Although the resistances and the reactances of a series circuit may be added directly to give, respectively, the resultant resistance and reactance, impedances may not be so added. Impedances must always be added in complex. For example,

$$\bar{z}_0 = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_k$$

can mean only

$$\bar{z}_0 = (r_1 + r_2 + \cdots + r_k) + j(x_1 + x_2 + \cdots + x_k) \tag{32}$$

It can never mean anything else. The resultant impedance in ohms is

$$z_0 = \sqrt{(r_1 + r_2 + \cdots + r_k)^2 + (x_1 + x_2 + \cdots + x_k)^2} \tag{33}$$
$$= \sqrt{r_0^2 + x_0^2} \tag{34}$$

where the r's and x's are expressed in ohms.

When only the ampere value of the current in a series circuit is desired, it is found by dividing the magnitude of the impressed voltage in volts by the magnitude of the resultant impedance in ohms.

$$\frac{V_0 \text{ (volts)}}{z_0 \text{ (ohms)}} = I_0 \text{ (amperes)} \tag{35}$$

The phase of the current with respect to the voltage is determined by equations (29), (30) and (31), in which θ_0 is the angle of lead of the voltage with respect to the current. The angle of lead of the current with respect to the voltage is $-\theta_0$. Since a negative angle of lead is equivalent to an angle of lag, the angle θ_0 in equations (29), (30) and (31) may equally well be considered as the angle of lag of the current with respect to the voltage.

Whether θ_0 actually represents an angle of lead or an angle of lag in any particular case depends upon its sign as fixed by the resultant reactance of the circuit. When θ_0, as determined by the resultant reactance x_0, is positive, *i.e.*, when the sum of the inductive reactances $\Sigma x_L = \Sigma 2\pi fL$ is greater than the sum of the capacitive reactances $\Sigma(x_C) = \Sigma\frac{-1}{2\pi fC}$, θ_0 is actually an angle of lead of the voltage with respect to the current or an angle of lag of the current with respect to the voltage. In this case, the volt-

age leads the current or the current lags the voltage. The circuit as a whole is inductive. When θ_0 is negative, it is actually an angle of lag of the voltage with respect to the current or an angle of lead of the current with respect to the voltage. In this case, the voltage lags the current or the current leads the voltage. The circuit as a whole is capacitive.

Example of Impedances in Series.—A circuit, which consists of two impedances and a non-inductive resistance in series, is connected across a 230-volt, 60-cycle circuit. One of the impedances has a self-inductance $L_1 = 0.1$ henry and a resistance $r_1 = 5$ ohms. The other impedance has a capacitance $C = 100$ microfarads and a negligible resistance. The non-inductive resistance is $r_3 = 10$ ohms. What is the current and what is its phase with respect to the impressed voltage? How much power does the circuit absorb and what is the power factor? What are the potential drops across each of the impedances and the non-inductive resistance, and what are the phase angles of these drops with respect to the current in the circuit? What are their phase angles with respect to the impressed voltage?

$$
\begin{aligned}
r_1 &= 5 \text{ ohms} \\
x_1 &= 2\pi 60 \times 0.1 = 377 \times 0.1 = 37.7 \text{ ohms} \\
r_2 &= 0 \\
x_2 &= \frac{-10^6}{2\pi 60 \times 100} = -26.53 \text{ ohms} \\
r_3 &= 10 \text{ ohms} \\
x_3 &= 0 \\
\bar{z}_0 &= (r_1 + r_2 + r_3) + j(x_1 + x_2 + x_3) \\
&= (5 + 0 + 10) + j(37.7 - 26.53 + 0) \\
&= 15 + j11.17 \\
z_0 &= \sqrt{(15)^2 + (11.17)^2} \\
&= 18.70 \text{ ohms} \\
I_0 &= \frac{230}{18.70} = 12.30 \text{ amperes}
\end{aligned}
$$

$$\text{Power factor} = \cos\theta_0 = \frac{r_0}{z_0} = \frac{15}{18.70} = 0.8021$$

$$\theta_0 = 36.67 \text{ degrees}$$

$$
\begin{aligned}
\text{Total power} = P &= 230 \times 12.30 \times 0.8021 \\
&= 2264 \text{ watts}
\end{aligned}
$$

The voltage impressed on the circuit leads the current by 36.67 degrees, or the current lags the voltage by the same angle. The circuit as a whole is inductive, since x_0 is positive.

$$\begin{aligned} E_1 \text{ (drop across } z_1) &= I_0 z_1 \\ &= 12.30 \times \sqrt{(5)^2 + (37.7)^2} \\ &= 12.30 \times 38.03 \\ &= 467.8 \text{ volts} \\ \tan \theta_1 &= \frac{37.7}{5} = 7.54 \\ \theta_1 &= 82.45 \text{ degrees} \end{aligned}$$

The voltage drop $\bar{E}_1$, across the inductive impedance, leads the current by 82.45 degrees.

$$\begin{aligned} E_2 \text{ (drop across } z_2) &= I_0 z_2 \\ &= 12.30 \times \sqrt{(0)^2 + (26.53)^2} \\ &= 12.30 \times 26.53 \\ &= 326.3 \text{ volts} \\ \tan \theta_2 &= \frac{-26.53}{0} = -\infty \\ \theta_2 &= -90 \text{ degrees} \end{aligned}$$

The voltage drop $\bar{E}_2$, across the capacitive impedance, lags the current by 90 degrees.

$$\begin{aligned} E_3 \text{ (drop across } z_3) &= I_0 z_3 \\ &= 12.30 \times \sqrt{(10)^2 + (0)^2} \\ &= 12.30 \times 10 \\ &= 123.0 \text{ volts} \\ \tan \theta_3 &= \frac{0}{10} = 0 \\ \theta_3 &= 0 \text{ degrees} \end{aligned}$$

The voltage drop $\bar{E}_3$, across the non-inductive resistance, is in phase with the current.

Since the voltage impressed across the circuit leads the current by 36.67 degrees, to get the phase of the voltage drops across the impedances and the non-inductive resistance with respect to the impressed voltage $\bar{V}_0$, 36.67 degrees must be subtracted from the phase angles of the drops with respect to the current.

Therefore, the phase angles of the drops with respect to the voltage are

$$\begin{aligned}
\theta_1' &= 82.45 - 36.67 = 45.78 \text{ degrees} \\
\theta_2' &= -90.0 - 36.67 = -126.7 \text{ degrees} \\
\theta_3' &= 0.0 - 36.67 = -36.67 \text{ degrees}
\end{aligned}$$

$\bar{E}_1$ leads the voltage drop $\bar{V}_0$ by 45.78 degrees.

$\bar{E}_2$ leads the voltage drop $\bar{V}_0$ by -126.7 degrees or lags it by 126.7 degrees.

$\bar{E}_3$ leads the voltage drop $\bar{V}_0$ by -36.67 degrees or lags it by 36.67 degrees.

It should be noted that the voltage drop across each impedance is much greater than the voltage drop across the entire circuit. This is always possible when a circuit contains both inductance and capacitance in series with a low resistance. In this particular problem, the drops across the two impedances are $82.45 + 90.0 = 172.45$ degrees apart in phase and, therefore, contribute little to the resultant voltage across the circuit More will be said about this condition under "Resonance."

The preceding problem could have been solved by the complex method, but in this particular problem the ordinary algebraic method gives a somewhat shorter solution.

Another Example of Impedances in Series.—A certain circuit consisting of two impedances, one of which has a resistance of 10 ohms and an inductive reactance of 10 ohms, takes a leading current of 15 amperes at 0.8 power factor when connected across a 200-volt, 60-cycle circuit. What are the resistance and reactance of the second impedance?

The use of the complex method gives the more direct solution in this case. Call the known impedance z_1 and the unknown impedance z_2. Take the current as the axis of reference. Then,

$$\begin{aligned}
\bar{I}_0 &= 15(1 + j0) = 15 + j0 \\
\bar{V}_0 &= 200(\cos\theta_0 - j\sin\theta_0) \\
&= 200(0.8 - j0.6) \\
&= 160 - j120 \\
\bar{z}_0 = r_0 + jx_0 &= \frac{\bar{V}_0}{\bar{I}_0} = \frac{160 - j120}{15 + j0}
\end{aligned}$$

$$
\begin{aligned}
&= 10.67 - j8.0 \\
r_0 &= 10.67 \text{ ohms} \\
x_0 &= -8.0 \text{ ohms} \\
r_2 &= 10.67 - r_1 \\
&= 10.67 - 10.00 = 0.67 \text{ ohm} \\
x_2 &= -8.0 - x_1 \\
&= -8.0 - 10.0 \\
&= -18.0 \text{ ohms}
\end{aligned}
$$

The second impedance is therefore capacitive, since x_2 is negative. Its complex expression is

$$\bar{z}_2 = 0.67 - j18.0$$

It has a resistance of 0.67 ohm and a capacitive reactance of 18.0 ohms. If x_2 is expressed in ohms and C_2 in microfarads,

$$
\begin{aligned}
x_2 &= \frac{-10^6}{2\pi f C_2} \\
C_2 &= \frac{-10^6}{2\pi 60(-18.0)} \\
&= 147.4 \text{ microfarads}
\end{aligned}
$$

Series Resonance.—A series circuit containing inductance and capacitance is said to be in resonance when the algebraic sum of the inductive and capacitive reactances is equal to zero. For resonance, therefore,

$$x_0 = x_1 + x_2 + x_3 + \cdots + x_k = 0 \tag{36}$$
$$= \Sigma x_L + \Sigma x_C = 0 \tag{37}$$

Consider a circuit containing a resistance r, an inductance L and a capacitance C, in series.

$$I = \frac{E}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

For resonance,

$$\omega L = \frac{1}{\omega C} \text{ or } \omega^2 LC = 1 \tag{38}$$

and

$$f = \frac{1}{2\pi\sqrt{LC}} \tag{39}$$

With constant frequency, resonance may be produced by adjusting either L or C or both, equation (38), or with L and C constant it may be produced by adjusting the frequency, equation (39). Since the resultant reactance is zero, the current at resonance is in phase with the voltage impressed across the entire circuit and follows Ohm's law. It is given by

$$I = \frac{E}{r}$$

At resonance, the resultant impedance of a series circuit is equal to its total resistance. If the resistance is constant, the impedance is a minimum at resonance. Under this condition and with fixed impressed voltage, the current is, therefore, a maximum and is limited only by the resistance of the circuit. It is entirely independent of the magnitudes of the inductive and capacitive reactances

The drop in potential due to the inductance is $I\omega L$. Due to the capacitance it is $\frac{-I}{\omega C}$. These two potential drops at resonance are without effect on the resultant potential drop across the circuit as a whole. They may be much greater than the voltage impressed on the entire circuit and may easily reach excessive values when the resultant resistance of the circuit is small in comparison with its inductive and capacitive reactances. Series resonance, for constant-potential circuits, is usually a very undesirable condition, on account of the excessive current and high voltages that may be produced when the resistance is small compared with the inductive and capacitive reactances.

If a low-resistance series circuit is tuned for resonance by varying the capacitance, and curves of current and of voltage drops across the inductance and capacitance are plotted against capacitance, they all show marked peaks at resonance. The maximum current and maximum voltage drop across the inductance occur at resonance, but the maximum voltage drop across the capacitance does *not* occur at resonance. If the capacitance is constant and the inductance is varied, the maximum drop across the capacitance then occurs at resonance, but now the maximum voltage drop across the inductance does *not* occur at resonance. The steepness with which the curves rise, as

resonance is approached, depends upon the magnitude of the resistance of the circuit compared with the magnitudes of the inductive and capacitive reactances. The point of resonance is most pronounced when the resistance is low in comparison with the reactances.

When a series circuit is tuned for resonance by varying either L, C or f, low resistance permits very sharp tuning. With high resistance, the resonance curves become flattened and sharp tuning is then impossible. With high resistance compared with

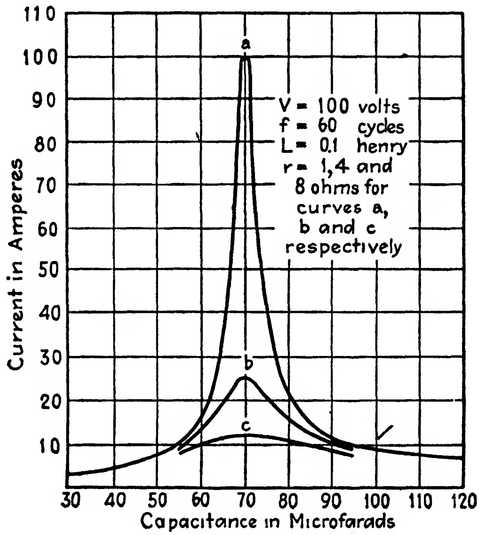


FIG. 63.

the reactances, the point of resonance is not well defined. This is illustrated in Fig. 63.

Three current curves, showing series resonance, are plotted in Fig. 63 for a 60-cycle, 100-volt circuit having an inductance of 0.1 henry and a variable capacitance. Curve a is for a resistance of 1 ohm. Curves b and c are for resistances of 4 and 8 ohms, respectively. The lower parts of the three curves so nearly coincide that for b and c only the portions of the curves near resonance are plotted. For curve a, the voltage drop across the inductance is 3770 volts at resonance. It is one-quarter and one-eighth this amount at resonance for curves b and c, respectively.

Figure 64 shows the effect of varying the frequency to produce resonance in a circuit containing constant resistance, constant inductance and constant capacitance in series. Curves of inductive reactance $x_L = \omega L$, capacitive reactance $x_C = \dfrac{-1}{\omega C}$,

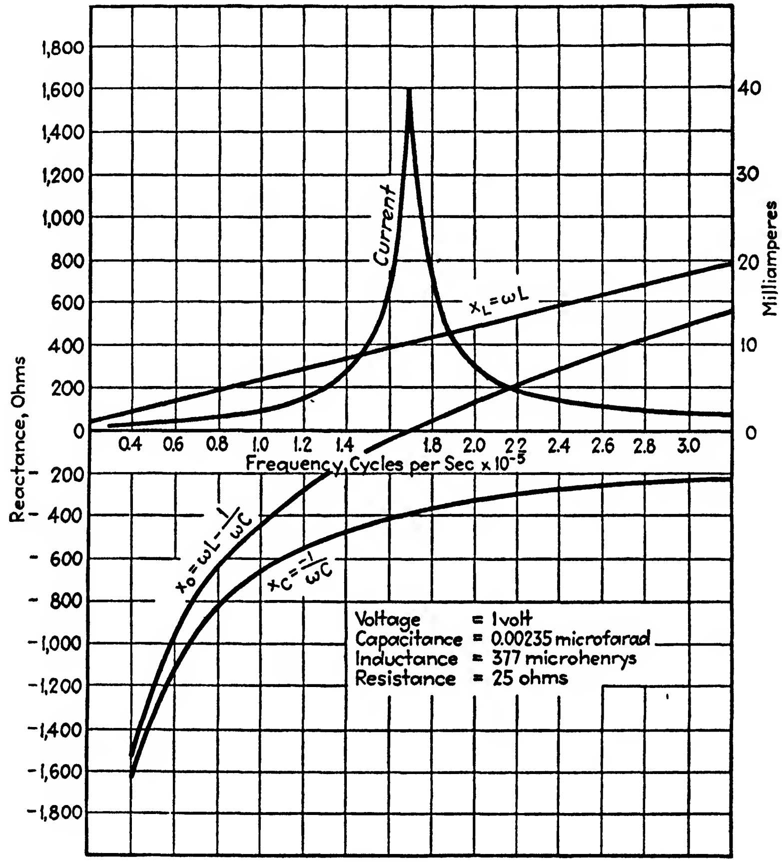


Fig. 64.

resultant reactance $x_L + x_C = \omega L - \dfrac{1}{C\omega}$ and current are plotted for a circuit having $r = 25$ ohms, $L = 377$ microhenrys and $C = 0.00235$ microfarad. Resonance occurs at a frequency of 1.69×10^5 cycles.

Free Period of Oscillation of a Resonant Circuit Containing Constant Resistance, Constant Inductance and Constant Capacitance in Series.—The natural or free oscillation frequency of a circuit containing constant resistance, constant inductance and constant capacitance in series is

$$f = \frac{\sqrt{1 - \frac{r^2C}{4L}}}{2\pi\sqrt{LC}} \tag{40}$$

[See equation (117), page 170.] When r^2 is small compared with $\frac{4L}{C}$, equation (40) reduces to

$$f = \frac{1}{2\pi\sqrt{LC}} \tag{41}$$

Therefore, the resonant frequency of a series circuit, which has low resistance compared with the ratio of its inductance and capacitance, is the same as its free oscillation frequency. This makes it possible to tune a series circuit containing inductance and capacitance to have a sharp current maximum for a definite frequency.

A circuit on which a non-sinusoidal electromotive force is impressed may be in resonance for a certain harmonic and yet be very far from resonance for the fundamental or other harmonics. [See equation (39), page 222.] A circuit can be in resonance for only one frequency at the same time. By proper tuning, as by varying either the inductance or the capacitance or both, it is possible to exaggerate any harmonic in the current for which there is a corresponding harmonic in the impressed voltage. Tuning is used in radio telegraphy in order to make a receiving circuit respond to electromagnetic waves of a definite frequency. On account of the very high frequency used in radio communication, the resistance of radio circuits is usually very small in comparison with their reactances, making very sharp tuning possible.

A Problem in Resonance.—A series circuit, which consists of a non-inductive resistance of 5 ohms, an impedance of 1.5 ohms resistance and 0.1 henry inductance and a variable capacitance, is connected across a constant-potential, 220-volt, 60-cycle cir-

cuit. If the capacitance, starting with a value too small to produce resonance, is gradually increased, for what value of the capacitance does the voltage drop across it have the greatest value? For what value of the capacitance does the voltage drop across the inductance have the greatest value? As the capacitance is increased, which of the two voltage drops has its maximum value first?

$$x_L = 377 \times 0.1 = 37.7 \text{ ohms}$$

$$V_C = I\frac{-1}{\omega C} = Ix_C$$

$$V_L = I\omega L = Ix_L$$

where V_C and V_L are the potential drops across the capacitance and inductance, respectively.

$$I = \frac{V_0}{\sqrt{r_0^2 + (x_L + x_C)^2}}$$

$$V_C = Ix_C = \left\{\frac{V_0}{\sqrt{r_0^2 + (x_L + x_C)^2}}\right\}x_C$$

$$\frac{dV_C}{dx_C} = \frac{V_0\sqrt{r_0^2+(x_L+x_C)^2} - V_0x_C0.5[r_0^2+(x_L+x_C)^2]^{-\frac{1}{2}}2(x_L+x_C)}{r_0^2 + (x_L + x_C)^2}$$

$$= 0$$

$$r_0^2 + (x_L + x_C)^2 - x_C(x_L + x_C) = 0$$

$$r_0^2 + x_L^2 + 2x_Lx_C + x_C^2 - x_Lx_C - x_C^2 = 0$$

$$x_C = -\frac{r_0^2 + x_L^2}{x_L} \tag{42}$$

$$= -\left[\frac{(6.5)^2}{33.7} + 37.7\right] = -38.8 \text{ ohms}$$

$$C = -\frac{10^6}{\omega x_C} = \frac{-10^6}{377 \times (-38.8)}$$

$$= 68.4 \text{ microfarads}$$

The capacitance for maximum voltage drop across the capacitance is therefore 68.4 microfarads.

The maximum voltage drop across the capacitance is

$$V_C \text{ (maximum)} = \frac{220}{\sqrt{(6.5)^2 + (37.7 - 38.8)^2}} \times 38.8$$

$$= \frac{220}{6.593} \times 38.8$$

$$= 1293 \text{ volts}$$

Since L is constant, the voltage drop across the inductance must be a maximum when the current is a maximum. The current is a maximum at resonance. Therefore,

$$\begin{aligned} V_L \text{ (maximum)} &= \frac{V_0}{r_0} x_L \\ &= \frac{220}{6.5} \times 37.7 \\ &= 1276 \text{ volts} \end{aligned}$$

At resonance, V_C and V_L are equal in magnitude. V_C at resonance is, therefore, 1276 volts, which is less than its maximum value.

$$\begin{aligned} -x_C &= x_L \\ C \text{ (at resonance)} &= \frac{-10^6}{377 \times (-37.7)} \\ &= 70.3 \text{ microfarads} \end{aligned}$$

The voltage drop across the capacitance reaches its maximum before the voltage drop across the inductance reaches its maximum. As resonance is approached (C increasing, x_C decreasing), the current rises very rapidly for a small decrease in x_C. The increase in the current more than balances the decrease in x_C until the point of resonance is nearly reached. As resonance is approached, the current curve flattens out preparatory to decreasing after resonance is passed. At some point just before resonance is reached, the decrease in x_C just balances the increase in I. The maximum voltage drop across the capacitance occurs at this point. When the resistance is small in comparison with x_L, the maximum voltage drop across the capacitance occurs very near the point of resonance. When the resistance is large, the maximum voltage drop across the capacitance may occur quite a bit before resonance is reached.

In the problem just solved, what additional resistance is it necessary to place in series with the circuit in order that the maximum voltage drop across the capacitance may be limited to 500 volts?

The capacitive reactance for maximum voltage drop across the capacitance is [equation (42)]

$$x_C = -\frac{r_0^2 + x_L^2}{x_L}$$

The resultant reactance is

$$x_0 = x_L + x_C = x_L - \frac{r_0^2 + x_L^2}{x_L} = -\frac{r_0^2}{x_L}$$

The resultant impedance is

$$z_0 = \sqrt{r_0^2 + \left(-\frac{r_0^2}{x_L}\right)^2} = \frac{r_0}{x_L}\sqrt{r_0^2 + x_L^2}$$

The resultant impedance drop across the condenser is

$$\begin{aligned} V_C = Ix_C &= \frac{V_0}{z_0} \times x_C \\ &= \frac{220}{\frac{r_0}{x_L}\sqrt{r_0^2 + x_L^2}} \times \frac{r_0^2 + x_L^2}{x_L} = 500 \text{ volts} \end{aligned}$$

$$\frac{\sqrt{r_0^2 + x_L^2}}{r_0} = \frac{500}{220}$$

$$\begin{aligned} r_0 &= \frac{x_L}{\sqrt{\left(\frac{500}{220}\right)^2 - 1}} = \frac{37.7}{\sqrt{\left(\frac{500}{220}\right)^2 - 1}} \\ &= 18.5 \text{ ohms total resistance} \end{aligned}$$

Added resistance

$$\begin{aligned} &= 18.5 - 1.5 - 5.0 \\ &= 12.0 \text{ ohms} \end{aligned}$$

Impedances in Parallel.—The voltages impressed across the branches of a circuit consisting of a number of impedances in parallel are equal. The resultant current taken by the circuit is equal to the vector sum of the component currents taken by the branches. This may be expressed analytically as follows:

$$\bar{V}_0 = \bar{E}_1 = \bar{E}_2 = \bar{E}_3 = \text{etc.} \tag{43}$$

$$= \bar{I}_1\bar{z}_1 = \bar{I}_2\bar{z}_2 = \bar{I}_3\bar{z}_3 = \text{etc.} \tag{44}$$

$$\bar{I}_0 = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \text{etc.} \tag{45}$$

$$= \frac{\bar{V}_0}{\bar{z}_1} + \frac{\bar{V}_0}{\bar{z}_2} + \frac{\bar{V}_0}{\bar{z}_3} + \text{etc.} \tag{46}$$

$$= \bar{V}_0\left(\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \text{etc.}\right) \tag{47}$$

Consider a circuit consisting of two impedances, z_1 and z_2, in parallel as indicated in Fig. 65(*a*).

The impedance z_1 of branch 1 has a resistance r_1 and an inductive reactance $x_1 = \omega L_1$. The impedance z_2 of branch 2 has a resistance r_2 and a capacitive reactance $x_2 = \dfrac{-1}{\omega C_2}$. Since the impedances are in parallel, the same potential V_0 is impressed across their terminals.

The graphical construction for this circuit is shown in Fig. 65(*b*). $\bar{V}_0$ is the voltage drop impressed on the circuit and is also

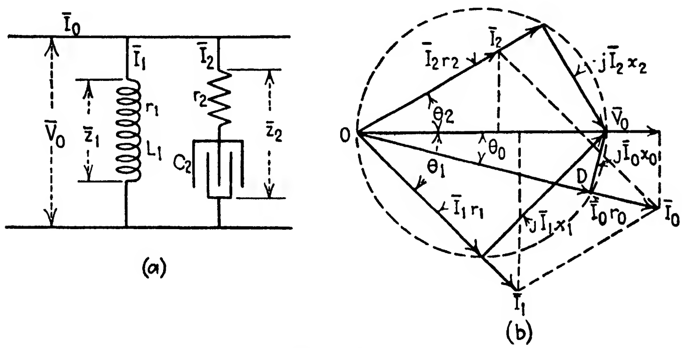


FIG. 65.

the potential drop across each of the branches. [Equation (43).] Since these drops are equal and in phase, $\bar{V}_0$ must be the common side of the right-angled voltage triangles. The intersection of the vectors representing the resistance and reactance drops in each of the branches of the circuit must, therefore, lie on a circle drawn with the vector $\bar{V}_0$ as diameter.

All voltages are drawn to a common scale. Likewise all currents are drawn to a common scale, but the scales for voltage and current may be and, indeed, usually are different.

The sides OD and DV_0 of the resultant triangle ODV_0 are the resultant resistance and reactance drops $r_0\bar{I}_0$ and $x_0\bar{I}_0$, respectively. Knowing $\bar{I}_0$, the resultant resistance r_0 and the resultant reactance x_0 may be found.

From the diagram, Fig. 65(*b*),

$$I_0 \cos \theta_0 = I_1 \cos \theta_1 + I_2 \cos \theta_2 \tag{48}$$
$$I_0 \sin \theta_0 = I_1 \sin \theta_1 + I_2 \sin \theta_2 \tag{49}$$

These equations may be written

$$\frac{V_0}{z_0} \times \frac{r_0}{z_0} = \frac{V_0}{z_1} \times \frac{r_1}{z_1} + \frac{V_0}{z_2} \times \frac{r_2}{z_2} \tag{50}$$

$$V_0 \times \frac{r_0}{r_0^2 + x_0^2} = V_0 \left(\frac{r_1}{r_1^2 + x_1^2} + \frac{r_2}{r_2^2 + x_2^2}\right) \tag{51}$$

$$\frac{V_0}{z_0} \times \frac{x_0}{z_0} = \frac{V_0}{z_1} \times \frac{x_1}{z_1} + \frac{V_0}{z_2} \times \frac{x_2}{z_2} \tag{52}$$

$$V_0 \times \frac{x_0}{r_0^2 + x_0^2} = V_0 \left(\frac{x_1}{r_1^2 + x_1^2} + \frac{x_2}{r_2^2 + x_2^2}\right) \tag{53}$$

Hence,

$$\frac{r_0}{r_0^2 + x_0^2} = \frac{r_1}{r_1^2 + x_1^2} + \frac{r_2}{r_2^2 + x_2^2} \tag{54}$$

$$\frac{x_0}{r_0^2 + x_0^2} = \frac{x_1}{r_1^2 + x_1^2} + \frac{x_2}{r_2^2 + x_2^2} \tag{55}$$

In general, if there are k branches in parallel,

$$\frac{r_0}{r_0^2 + x_0^2} = \Sigma_1^k \frac{r}{r^2 + x^2} \tag{56}$$

$$\frac{x_0}{r_0^2 + x_0^2} = \Sigma_1^k \frac{x}{r^2 + x^2} \tag{57}$$

It must be remembered that x is positive for inductance and negative for capacitance. Therefore, the summations in equations (49), (52), (53), (55) and (57) must be made in an algebraic sense.

Conductance, Susceptance and Admittance of a Circuit.—The expression

$$\frac{r}{r^2 + x^2} = g \tag{58}$$

is known as the *conductance of a circuit* and is denoted by the letter g. When there are a number of circuits in parallel, their resultant conductance, from equation (56), is given by the sum of their separate conductances. Therefore,

$$\begin{aligned} g_0 &= g_1 + g_2 + \cdots + g_k \\ &= \Sigma_1^k g \end{aligned} \tag{59}$$

The expression

$$\frac{x}{r^2 + x^2} = b \tag{60}$$

is known as the *susceptance of a circuit* and is denoted by the letter b. When there are a number of circuits in parallel, their resultant susceptance, from equation (57), is given by the sum of their separate susceptances. Since reactance x may be either positive or negative, according to whether it is inductive or capacitive reactance, it is obvious from the expression for susceptance, equation (60), that susceptance is likewise either positive or negative. Inductive susceptance is positive. Capacitive susceptance is negative. Since susceptance may be either positive or negative, susceptances must always be added in an algebraic sense, *i.e.*, with regard to their signs.

$$\begin{aligned} b_0 &= b_1 + b_2 + \cdot \cdot \cdot + b_k \\ &= \Sigma_1^k b \end{aligned} \tag{61}$$

Since $I_0 \cos \theta_0 = V_0 g_0$ and $I_0 \sin \theta_0 = V_0 b_0$ are two quadrature components of the resultant current I_0 [see Fig. 65(b)], it is obvious that

$$I_0 = V_0\sqrt{g_0^2 + b_0^2} \tag{62}$$

$V_0 g_0$ and $V_0 b_0$ are the active and reactive components of the current I_0 with respect to the voltage V_0.

The expression

$$y_0 = \sqrt{g_0^2 + b_0^2} \tag{63}$$

is known as the *admittance of the circuit*. Admittance is denoted by the letter y. From equations (62) and (63),

$$y_0 = \frac{I_0}{V_0} \tag{64}$$

Impedance is $z_0 = \dfrac{V_0}{I_0}$. Admittance, therefore, is the reciprocal of impedance. Although admittance is always the reciprocal of impedance, conductance is never the reciprocal of resistance except when the reactance is zero. [Equation (58).] Susceptance is never the reciprocal of reactance except when the resistance is zero. [Equation (60).]

Admittance and also conductance and susceptance are measured in reciprocal ohms. This unit is called the *mho*, the word ohm written backward. When resistance and reactance in equations (58) and (60) are expressed in ohms, the conductance and susceptance are in mhos. When I_0 and V_0 in equation (64) are in amperes and volts, the admittance y_0 is in mhos.

The current in a series circuit in amperes is given by the resultant impressed voltage in volts divided by the resultant impedance of the circuit in ohms. The resultant current in a parallel circuit in amperes is given by the impressed voltage in volts multiplied by the resultant admittance of the circuit in mhos. For the parallel circuit,

$$\tan \theta_0 = \frac{b_0}{g_0}, \quad \cos \theta_0 = \frac{g_0}{y_0}, \quad \sin \theta_0 = \frac{b_0}{y_0} \tag{65}$$

Vector Method.—From equation (47), page 229,

$$\bar{I}_0 = \bar{V}_0\left(\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \text{etc.}\right) \tag{66}$$

Putting the z's in their complex form, equation (66) becomes

$$\bar{I}_0 = \bar{V}_0\left(\frac{1}{r_1 + jx_1} + \frac{1}{r_2 + jx_2} + \frac{1}{r_3 + jx_3} + \text{etc.}\right) \tag{67}$$

Rationalizing,

$$\begin{aligned}\bar{I}_0 = \bar{V}_0\Big(\frac{1}{r_1 + jx_1} \times \frac{r_1 - jx_1}{r_1 - jx_1} + \frac{1}{r_2 + jx_2} \times \frac{r_2 - jx_2}{r_2 - jx_2} \\ + \frac{1}{r_3 + jx_3} \times \frac{r_3 - jx_3}{r_3 - jx_3} + \text{etc.}\Big)\end{aligned}$$

$$\begin{aligned}= \bar{V}_0\Big\{\left(\frac{r_1}{r_1^2 + x_1^2} - j\frac{x_1}{r_1^2 + x_1^2}\right) + \left(\frac{r_2}{r_2^2 + x_2^2} - j\frac{x_2}{r_2^2 + x_2^2}\right) \\ + \left(\frac{r_3}{r_3^2 + x_3^2} - j\frac{x_3}{r_3^2 + x_3^2}\right) + \text{etc.}\Big\}\end{aligned} \tag{68}$$

$$= \bar{V}_0\{(g_1 + g_2 + g_3 + \text{etc.}) - j(b_1 + b_2 + b_3 + \text{etc.})\} \tag{69}$$

$$g_0 = g_1 + g_2 + g_3 + \text{etc.} \tag{70}$$

$$b_0 = b_1 + b_2 + b_3 + \text{etc.} \tag{71}$$

It should be noted that impedance is $\bar{z} = r + jx$ and admittance is $\bar{y} = g - jb$. Since inductive reactance is positive and capacitive reactance is negative, the impedance of an inductive

circuit is actually $\bar{z} = r + jx$ and the impedance of a capacitive circuit *is actually* $\bar{z} = r - jx$. *Since inductive susceptance is* positive and capacitive susceptance is negative, the admittance of an inductive circuit is actually $\bar{y} = g - jb$ and the admittance of a capacitive circuit is actually $\bar{y} = g + jb$. The signs of the second terms in the complex expressions for impedance and admittance are opposite.

For impedances in parallel, the resultant conductance is the sum of the component conductances and the resultant susceptance is the algebraic sum of the component susceptances. Conductances, susceptances and admittances, as has already been stated, are measured in reciprocal ohms, *i.e.*, in mhos.

Admittances cannot be added directly. Admittance is a complex operator. Admittances, therefore, like impedances, can be added only when expressed in complex form. For example:

$$y_0 = y_1 + y_2 + y_3 + \cdots + y_k$$

can mean only

$$\begin{aligned}\bar{y}_0 &= \bar{y}_1 + \bar{y}_2 + \bar{y}_3 + \cdots + \bar{y}_k \\ &= (g_1 + g_2 + g_3 + \cdots + g_k) \\ &\quad -j(b_1 + b_2 + b_3 + \cdots + b_k)\end{aligned} \tag{72}$$

It can never mean anything else.

Resistance and Reactance in Terms of Conductance and Susceptance.—It is often necessary, when dealing with a parallel circuit, to determine its resultant resistance and reactance, *i.e.*, the resistance and reactance of the simple series circuit which would replace it so far as total power, resultant current and power factor are concerned. The expressions for r and x in terms of g and b may be found from the relation between impedance and admittance

$$\begin{aligned}\bar{z} = r + jx &= \frac{1}{\bar{y}} = \frac{1}{g - jb} \\ &= \frac{1}{g - jb} \times \frac{g + jb}{g + jb} \\ &= \frac{g}{g^2 + b^2} + j\frac{b}{g^2 + b^2}\end{aligned} \tag{73}$$

Evidently,

$$r = \frac{g}{g^2 + b^2} \tag{74}$$

$$x = \frac{b}{g^2 + b^2} \tag{75}$$

Polar Expression for Admittance.—It is often convenient, when dealing with the more complicated problems, such as those associated with the transmission line, to use the polar expression for admittance. The polar expressions for impedance are given in Chapter V, pages 145, 160 and 179.

$$\begin{aligned}
\bar{y} &= g - jb \\
&= (g - jb) \times \frac{\sqrt{g^2 + b^2}}{\sqrt{g^2 + b^2}} \\
&= \sqrt{g^2 + b^2}\left(\frac{g}{\sqrt{g^2 + b^2}} - j\frac{b}{\sqrt{g^2 + b^2}}\right) \\
&= y(\cos\theta - j\sin\theta) \qquad (76) \\
&= y\underline{|-\theta} = y\overline{|\theta} \qquad (77)
\end{aligned}$$

The expression $y\underline{|-\theta}$ or $y\overline{|\theta}$ is known as the *polar expression for admittance.* The angle θ is the angle of the admittance and is determined by the relation

$$\theta = \tan^{-1}\frac{b}{g} \tag{78}$$

Admittance is a scalar quantity y, multiplied by an operator $\underline{|-\theta} = (\cos\theta - j\sin\theta)$, which rotates through an angle $-\theta$. When vector voltage is multiplied by admittance in either complex or polar form, it gives vector current in its proper phase relation with respect to the voltage, regardless of the axis to which the voltage is referred.

An Example of the Solution of a Parallel Circuit by the Use of Conductance, Susceptance and Admittance.—An impedance coil of 10 ohms resistance and 0.1 henry inductance, a condenser of negligible resistance and 100 microfarads capacitance and a non-inductive resistance of 20 ohms are connected in parallel across a 200-volt, 60-cycle circuit. What are the resultant current and power taken by the impedance coil, condenser and

resistance in parallel? What is the resultant power factor of the circuit? What current and power does each branch of the circuit take? What is the power factor of each branch?

$$\omega = 2\pi f = 377$$
$$\bar{z}_1 = 10 + j377 \times 0.1$$
$$= 10 + j37.7 \text{ ohms}$$
$$\bar{z}_2 = 0 - j\frac{10^6}{377 \times 100}$$
$$= 0 - j26.53 \text{ ohms}$$
$$\bar{z}_3 = 20 + j0 \text{ ohms}$$
$$\bar{y}_1 = \frac{1}{\bar{z}_1} = \frac{10}{(10)^2 + (37.7)^2} - j\frac{37.7}{(10)^2 + (37.7)^2}$$
$$= 0.006575 - j0.02479 \text{ mho}$$
$$\bar{y}_2 = \frac{1}{\bar{z}_2} = 0 + j\frac{26.53}{(0)^2 + (26.53)^2}$$
$$= 0 + j0.03770 \text{ mho}$$
$$\bar{y}_3 = \frac{1}{\bar{z}_3} = \frac{20}{(20)^2 + (0)^2} - j\frac{0}{(20)^2 + (0)^2}$$
$$= 0.0500 - j0 \text{ mho}$$
$$\bar{y}_0 = \Sigma_1^3 g - j\Sigma_1^3 b$$
$$= 0.05658 + j0.01291 \text{ mho}$$

Take $\bar{V}_0$, the voltage drop impressed on the circuit, as the axis of reals. Then,

$$\bar{V}_0 = 200(1 + j0)$$
$$\bar{I}_0 = \bar{V}_0 \bar{y}_0$$
$$= 200(1 + j0)(0.05658 + j0.01291)$$
$$= 11.32 + j2.582$$
$$I_0 = \sqrt{(11.32)^2 + (2.582)^2}$$
$$= 11.61 \text{ amperes}$$
$$P_0 = V_0 I_0 \cos \theta_0$$
$$= V_0(V_0 y_0)\frac{g_0}{y_0}$$
$$= V_0^2 g_0$$
$$= (200)^2 \times (0.05658)$$
$$= 2263 \text{ watts}$$
$$\cos \theta_0 = \frac{V_0 g_0}{V_0 y_0}$$

$$= \frac{11.32}{11.61} = 0.9746 = \text{power factor for the circuit}$$

$$\theta_0 = 12.95 \text{ degrees}$$

Since $\tan \theta_0 = \dfrac{b_0}{g_0}$ is negative (b_0 is negative), θ_0 is negative and the resultant current $\bar{I}_0$ leads the voltage $\bar{V}_0$ impressed on the circuit by 12.95 degrees.

$$\begin{aligned}
\bar{I}_1 &= \bar{V}_0 \bar{y}_1 \\
&= 200(1 + j0)(0.006575 - j0.02479) \\
&= 1.315 - j4.958 \\
I_1 &= \sqrt{(1.315)^2 + (4.958)^2} \\
&= 5.13 \text{ amperes} \\
P_1 &= V_0{}^2 g_1 \\
&= (200)^2 \times (0.006575) \\
&= 263 \text{ watts} \\
\cos \theta_1 &= \frac{V_0 g_1}{V_0 y_1} \\
&= \frac{1.315}{5.13} = 0.2563 = \text{power factor for branch 1} \\
\theta_1 &= 75.15 \text{ degrees}
\end{aligned}$$

Since $\tan \theta_1 = \dfrac{b_1}{g_1}$ is positive, θ_1 is positive and the current $\bar{I}_1$ lags the voltage $\bar{V}_0$ impressed on the circuit by 75.15 degrees.

$$\begin{aligned}
\bar{I}_2 &= \bar{V}_0 \bar{y}_2 \\
&= 200(1 + j0)(0 + j0.03770) \\
&= 0 + j7.540 \\
I_2 &= \sqrt{(0)^2 + (7.540)^2} \\
&= 7.54 \text{ amperes} \\
P_2 &= V_0{}^2 g_2 \\
&= (200)^2 \times (0) \\
&= 0 \text{ watts} \\
\cos \theta_2 &= \frac{V_0 g_2}{V_0 y_2} \\
&= \frac{0.0}{7.540} = 0.0 = \text{power factor for branch 2} \\
\theta_2 &= 90 \text{ degrees}
\end{aligned}$$

Since $\tan \theta_2 = \dfrac{b_2}{g_2}$ is negative, θ_2 is negative and the current $\bar{I}_2$ leads the voltage $\bar{V}_0$ impressed on the circuit by 90 degrees.

$$\begin{aligned}
\bar{I}_3 &= \bar{V}_0\,\bar{y}_3 \\
&= 200(1 + j0)(0.0500 - j0) \\
&= 10.00 - j0 \\
I_3 &= \sqrt{(10.00)^2 + (0)^2} \\
&= 10.00 \text{ amperes} \\
P_3 &= V_0{}^2 g_3 \\
&= (200)^2 \times (0.0500) \\
&= 2000 \text{ watts} \\
\cos\theta_3 &= \frac{V_0\,g_3}{V_0\,y_3} \\
&= \frac{10.00}{10.00} = 1.0 = \text{power factor for branch 3} \\
\theta_3 &= 0 \text{ degrees}
\end{aligned}$$

The current $\bar{I}_3$ is in phase with the voltage $\bar{V}_0$ impressed on the circuit.

$$\begin{aligned}
P_0 &= P_1 + P_2 + P_3 \\
&= 263 + 0 + 2000 \\
&= 2263 \text{ watts}
\end{aligned}$$

Parallel Resonance.—A parallel circuit is said to be in resonance when its resultant susceptance is zero.

$$\begin{aligned}
\bar{I}_0 &= \bar{V}_0\,\bar{y}_0 \\
&= \bar{V}_0(g_0 - jb_0) \qquad (79)
\end{aligned}$$

$$\cos\theta_0 = \frac{g_0}{\sqrt{g_0{}^2 + b_0{}^2}} \qquad (80)$$

At resonance,

$$b_0 = 0$$

and

$$\bar{I}_0 = \bar{V}_0\,g_0 \qquad (81)$$

$$\cos\theta_0 = 1 \qquad (82)$$

For fixed resultant conductance, the resultant admittance of a parallel circuit is a minimum at resonance, being equal to the resultant conductance. The resultant current is also a minimum. Consequently, the resultant impedance is a maximum and would

be infinite if it were possible to have the resultant conductance zero. For the resultant conductance to be zero, the resistances of all parallel branches of the circuit would have to be zero or infinite. For a series circuit with fixed resistance, the resultant impedance is a minimum at resonance and is equal to the resultant resistance. For both series and parallel circuits, the power factor is unity at resonance and the resultant current is in phase with the voltage. For a series circuit in resonance, the sum of the reactive components of the voltage drops across the component parts of the circuit is zero. For a parallel circuit in resonance, the sum of the reactive components of the currents in the parallel branches of the circuit is zero.

For a parallel circuit with constant impressed voltage, there can be no rise in voltage across any branch of the circuit at resonance. Although the resultant current is a minimum, the currents in the inductive and capacitive branches may be large if their impedances are small. Since the impedance is a maximum for parallel resonance, parallel resonance for a constant-current circuit, *i.e.*, for a circuit in which the current is maintained constant, would be undesirable in most cases, since the potential drop $V_0 = \frac{I_0}{y_0}$ across it at resonance would in general be excessive. The currents in the inductive and capacitive branches probably would also be excessive.

In general, parallel resonance is a highly desirable condition for a constant-potential power circuit, since it gives a minimum resultant line current for a given amount of power transmitted. All commercial power circuits, except those for street lighting with series arc or series incandescent lamps, are operated at constant potential and the loads are in parallel. A few series power circuits, other than those for street lighting, are used abroad, but these are all direct-current circuits. Parallel resonance not only permits the power to be transmitted with a minimum line loss, *i.e.*, with a minimum I^2r loss in the line, but, what is more important, it also permits the use of the minimum generator capacity for a fixed amount of power transmitted. The output of alternating-current generators is limited by the product of current and voltage, *i.e.*, by volt-amperes and not by watts. Their power output is, therefore, a maximum at unity

power factor, which corresponds to resonance. Special devices are used in many cases to raise the power factors of commercial power circuits by producing resonance or partial resonance by placing capacitive loads in parallel with the existing commercial inductive loads. The saving in generator and transformer capacity, line-copper and line-transmission losses in many cases much more than balances the cost of the apparatus for correcting the power factor. The saving in space occupied by the generating and distributing apparatus is an important factor.

Series resonance would be a highly desirable condition for a constant-current power circuit, since it would make the resultant voltage impressed across the circuit a minimum for a fixed amount of power transmitted. With the current maintained constant, there would be little danger of excessive voltage drops across the component parts of the circuit.

Figure 66 shows the effect of varying the frequency to produce parallel resonance in a circuit consisting of two branches in parallel, one containing an inductance, $L = 377$ microhenrys, in series with a resistance, $r_L = 10$ ohms, the other containing a capacitance, $C = 0.00235$ microfarad, in series with a resistance, $r_C = 10$ ohms. The impressed voltage is 1 volt. The admittance, which owing to the low resistances is substantially equal to the resultant susceptance, and also the current are a minimum at resonance, which occurs at about 1.7×10^5 cycles. The constants of the circuit for which Fig. 66 is plotted have magnitudes such as might occur in radio work.

The curve of b_c plotted against frequency is not actually a straight line as it appears to be in Fig. 66. In this figure, it is a straight line to within the accuracy of the plot, because of the small values of r_C compared with x_C within the range of frequencies shown.

The curves in Fig. 66 should be compared with those for series resonance in Fig. 64, page 225.

Although minimum resultant current occurs in a constant-potential parallel circuit at resonance when the conductances of the parallel branches are constant, minimum resultant current does not occur at resonance if the resistances instead of the conductances of the parallel branches are constant. Since conductance is a function of reactance as well as of resistance,

when the resistances of the parallel branches are constant the conductances of these branches do not remain constant when their reactances are varied to produce resonance.

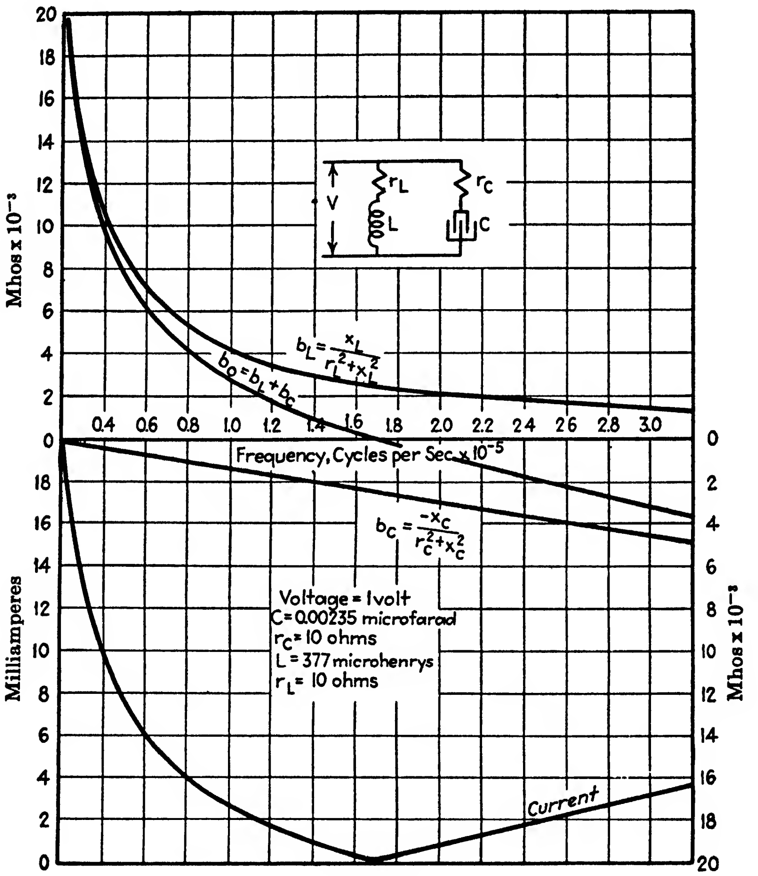


FIG. 66.

Consider a circuit with two parallel branches on which a constant voltage at constant frequency is impressed. Let one branch consist of a constant resistance in series with a constant inductance. Let the other branch contain a constant resistance in series with a variable capacitance. The diagram of connections of the circuit is shown in Fig. 67.

Refer to Fig. 68. The current I_L in the inductive branch is fixed in direction and magnitude with respect to the voltage V. When the capacitance of the condenser is infinite, x_c is zero and the current I_c in the capacitive branch is in phase with the impressed voltage V. Under this condition,

$$V = I_c\sqrt{r_c^2 + x_c^2} = I_c r_c$$

If the capacitance is now decreased, I_c decreases and leads the voltage V which is still equal to $I_c\sqrt{r_c^2 + x_c^2}$, but in this

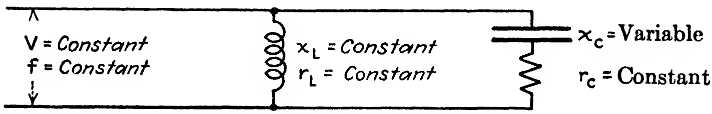


FIG. 67.

case x_c is not zero. As the capacitance is decreased, the end of the vector representing the $I_c r_c$ drop traces a semicircle with the impressed voltage V as a diameter. This semicircle is shown dotted in Fig. 68. Since r_c is constant, I_c must vary

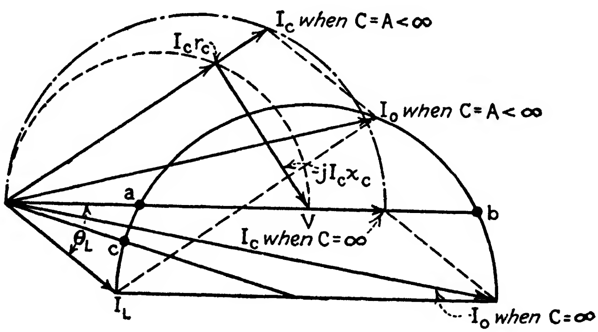


FIG. 68.

directly as $I_c r_c$. The end of I_c, therefore, also traces a semicircle. The diameter of this semicircle is the vector I_c for the condition $C = \infty$. This circle is shown by a dot-and-dash line. Since I_0 is found by adding to I_c a vector of fixed magnitude and fixed direction, if I_c travels on a semicircle, I_0 must also travel on a semicircle. This latter semicircle is shown

by a full line. Its diameter lies between the end of the vector I_0 for the condition $C = \infty$ and the end of the vector I_L.

Resonance occurs when the vector I_0 is in phase with V. As the capacitance of the condenser circuit is decreased from infinity, resonance first occurs when the end of I_0 lies at the point b. If the capacitance is further decreased, I_0 decreases and resonance again occurs when the end of I_0 lies at the point a. Neither of these resonant conditions corresponds to minimum I_0. Minimum I_0 does not occur until the capacitance is further decreased to the value which makes the vector I_0 normal to the semicircle *cab*, on which its end lies. This occurs when the end of I_0 lies at the point c.

When I_0 has its minimum value, the extension of the vector which represents it passes through the center of the semicircle *cab*. A line drawn between the end of the vector I_L and the point c is equal in magnitude and direction to the vector representing the current taken by the capacity branch when the resultant current I_0 is a minimum. The conditions which produce minimum resultant current can easily be calculated from Fig. 68.

Since the diameter of the circle on which I_0 swings is equal to $\frac{V}{r_c}$, when $I_L \sin \theta_L$ is equal to $\frac{V}{2r_c}$ this circle is tangent to the line *ab*, Fig. 68. Under this condition, there is only one value of the condenser capacitance which produces resonance. When $I_L \sin \theta_L$ is greater than $\frac{V}{r_c}$, resonance cannot be produced by any value of the condenser capacitance.

When resonance is produced in power circuits by use of an overexcited synchronous motor which takes a leading current, resonance corresponds to the condition of minimum resultant current, since the equivalent conductances of the loads are constant for any fixed loads. In radio circuits, resonance produces practically minimum current, since in high-frequency radio circuits the resistances are small compared with the reactances.

Example of Parallel Resonance.—An impedance of 0.1 henry inductance and 10 ohms resistance and a capacitance of 70 microfarads and negligible resistance are connected in parallel. For what frequency does the circuit act like a non-inductive resistance? What is the value of this resistance?

$$x_L = 2\pi f L = 6.284 \times f \times 0.1 = 0.6284f \text{ ohms}$$

$$x_C = \frac{-1}{2\pi f C} = \frac{-10^6}{6.284 \times f \times 70} = -2275\frac{1}{f} \text{ ohms}$$

For the circuit to act like a non-inductive resistance, the resultant susceptance of the circuit must be zero. For the susceptance to be zero, the circuit must be in resonance. For resonance,

$$\Sigma b = 0$$

$$\frac{x_L}{r_L^2 + x_L^2} = \frac{-x_C}{r_C^2 + x_C^2}$$

$$\frac{0.6284f}{(10)^2 + (0.6284f)^2} = \frac{2275\frac{1}{f}}{(0)^2 + \left(2275\frac{1}{f}\right)^2}$$

$$1430 = 100 + 0.3949f^2$$

$$f = \sqrt{\frac{1330}{0.3949}}$$

$$= 58.03 \text{ cycles}$$

$$x_L \text{ (at resonance)} = 2\pi \times 58.03 \times 0.1$$

$$= 36.46 \text{ ohms}$$

$$\bar{y}_0 = g_0 + jb_0$$

$$= \frac{r_L}{r_L^2 + x_L^2} + \frac{r_C}{r_C^2 + x_C^2} - j0$$

$$= \frac{10}{(10)^2 + (36.46)^2} + \frac{0}{(0)^2 + x_C^2} - j0$$

$$= 0.006994 - j0$$

$$\bar{z}_0 = \frac{1}{\bar{y}_0} = \frac{1}{0.006994 - j0}$$

$$= 143.0 + j0$$

$$r_0 = 143.0 \text{ ohms}$$

Impedances in Series Parallel.—When a circuit consists of a number of series elements, some of which are made up of two or more impedances in parallel, the elements containing the impedances in parallel must be replaced by their equivalent simple impedances by means of equations (74) and (75), page 235. The circuit may then be solved like a simple series circuit. An example will make this clear. Two impedances z_1 and z_2 in parallel are connected in series with a third impedance z_3. How

much power and current does the entire circuit take when it is connected across 200 volts? What are the currents in the impedances z_1 and z_2?

$$\bar{z}_1 = 5 + j2 \text{ ohms}$$
$$\bar{z}_2 = 6 - j4 \text{ ohms}$$
$$\bar{z}_3 = 1 + j3 \text{ ohms}$$

$$\bar{y}_1 = \frac{1}{\bar{z}_1} = \frac{r_1}{r_1^2 + x_1^2} - j\frac{x_1}{r_1^2 + x_1^2}$$
$$= \frac{5}{(5)^2 + (2)^2} - j\frac{2}{(5^2) + (2)^2}$$
$$= 0.1723 - j0.0690 \text{ mho}$$

$$\bar{y}_2 = \frac{1}{\bar{z}_2} = \frac{r_2}{r_2^2 + x_2^2} - j\frac{x_2}{r_2^2 + x_2^2}$$
$$= \frac{6}{(6)^2 + (4)^2} + j\frac{4}{(6)^2 + (4)^2}$$
$$= 0.1154 + j0.0769 \text{ mho}$$

The resultant admittance of z_1 and z_2 in parallel is

$$\bar{y}_{12} = \bar{y}_1 + \bar{y}_2$$
$$= 0.2877 + j0.0079$$
$$= 0.2877 - j(-0.0079) \text{ mho}$$

The general form for admittance is $\bar{y} = g - jb$. Therefore b_{12} is negative, as indicated.

The resultant impedance of the impedances z_1 and z_2 in parallel is

$$\bar{z}_{12} = \frac{1}{\bar{y}_{12}} = \frac{g_{12}}{(g_{12})^2 + (b_{12})^2} + j\frac{b_{12}}{(g_{12})^2 + (b_{12})^2}$$
$$= \frac{0.2877}{(0.2877)^2 + (0.0079)^2} + j\frac{-0.0079}{(0.2877)^2 + (0.0079)^2}$$
$$= 3.475 - j0.0954 \text{ ohms}$$
$$z_{12} = \sqrt{(3.475)^2 + (0.0954)^2}$$
$$= 3.476 \text{ ohms}$$

The resultant impedance of the entire circuit is

$$\bar{z}_0 = \bar{z}_3 + \bar{z}_{12}$$
$$= (1 + j3) + (3.475 - j0.0954)$$
$$= 4.475 + j\,2.905 \text{ ohms}$$

Take the impressed voltage drop $\bar{V}_0$ as the axis of reals. Then,

$$\begin{aligned}
\bar{V}_0 &= 200 + j0 \\
\bar{I}_0 &= \frac{\bar{V}_0}{\bar{z}_0} = \frac{200 + j0}{4.475 + j2.905} \\
&= \frac{200 + j0}{4.475 + j2.905} \times \frac{4.475 - j2.905}{4.475 - j2.905} \\
&= 31.45 - j20.42 \text{ amperes} \\
I_0 &= \sqrt{(31.45)^2 + (20.42)^2} \\
&= 37.5 \text{ amperes} \\
P_0 = \text{total power} &= 200 \times 31.45 + 0 \times (-20.42) \\
&= 6290 \text{ watts} \\
&= I_0{}^2 r_0 \\
&= (37.5)^2 \times 4.475 \\
&= 6290 \text{ watts}
\end{aligned}$$

The voltage drop across the impedances z_1 and z_2 in parallel is

$$\begin{aligned}
\bar{V}_{12} &= \bar{V}_0 - \bar{I}_0 \bar{z}_1 \\
&= (200 + j0) - (31.45 - j20.42)(1 + j3) \\
&= 107.3 - j73.93 \text{ volts} \\
V_{12} &= \sqrt{(107.3)^2 + (73.93)^2} \\
&= 130.4 \text{ volts} \\
&= I_0 z_{12} = 37.5 \times 3.476 = 130.4 \text{ volts} \\
I_1 &= \frac{V_{12}}{z_1} = \frac{130.4}{\sqrt{(5)^2 + (2)^2}} = \frac{130.4}{5.39} \\
&= 24.19 \text{ amperes} \\
I_2 &= \frac{V_{12}}{z_2} = \frac{130.4}{\sqrt{(6)^2 + (4)^2}} = \frac{130.4}{7.21} \\
&= 18.08 \text{ amperes}
\end{aligned}$$

The fact that series resonance gives minimum impedance for fixed total resistance, and parallel resonance gives maximum impedance for fixed total conductance has already been mentioned. This difference between the impedances of series and parallel circuits at resonance makes it possible to design a series-parallel circuit or network to respond to one frequency and at the same time to suppress another frequency. Such a circuit is known as a *filter circuit.* Filters are largely used in telephone circuits and radio circuits. For such purposes they are designed

to pass frequencies either above or below a certain value or to pass a band of frequencies. Filters are considered in Chapter VIII.

A simple coupled circuit which can be tuned to suppress one frequency and pass another is shown in Fig. 69. L_2 is an impedance. C_1 and C_3 are capacitances. D is a detector. Electric oscillations are impressed on the circuit at E. With fixed C_3, L_2 and C_1 are first adjusted for parallel resonance for the frequency to be suppressed. Their resultant parallel impedance for this frequency is, therefore, a maximum. Since the detector circuit is in series with L_2 and C_1 in parallel, little current of the frequency to be suppressed flows through the detector. C_3 is now adjusted to produce series resonance in the series circuit consisting of the detector and C_3 in series with L_2 and C_1 in parallel. This makes the impedance of the circuit as a whole a minimum for the frequency to be detected. By this double series-parallel tuning, minimum impedance is produced in the detector circuit for the frequency to be detected and maximum impedance is produced for the frequency to be suppressed.

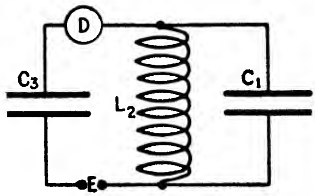


Fig. 69.

The above mentioned method of tuning assumes that the frequency to be suppressed is higher than the one to be detected. If the frequency to be detected were the higher, C_3 would have to be replaced by an inductance.

An Example of Series-parallel Tuning of a Circuit to Prevent Response to an Interfering Frequency.—The circuit shown in Fig. 69 is to be adjusted to respond to a frequency of 50,000 cycles and to damp out an interfering frequency of 100,000 cycles. L_2 is an impedance of 10,000 microhenrys self-inductance and 80 ohms resistance. D is a detector having negligible inductance and 15 ohms resistance.

The circuit consisting of L_2 and C_1 in parallel is first adjusted for parallel resonance for 100,000 cycles. Then, without changing L_2 or C_1, the capacitance C_3 is adjusted to produce series resonance in the circuit consisting of the detector D and the capacitance C_3 in series with the capacitance C_1 and the induc-

tance L_2 in parallel. The resistances of the capacitances are assumed to be negligible.

At 100,000 cycles,

$$\bar{z}_2 = 80 + j10^{-2} \times 2\pi \times 10^5$$
$$= 80 + j6280 \text{ ohms}$$

At 50,000 cycles,

$$\bar{z}_2 = 80 + j10^{-2} \times 2\pi \times 5 \times 10^4$$
$$= 80 + j3140 \text{ ohms}$$

For parallel resonance of L_2 and C_1 at 100,000 cycles, the resultant susceptance of the circuit consisting of L_2 and C_1 in parallel must be zero. Therefore, since the resistances of the capacitances are assumed to be negligible,

$$C_1 = \frac{L_2}{r_2^2 + \omega^2 L_2^2} \text{ farads}$$
$$= \frac{10^{-2}}{(80)^2 + (6280)^2} \text{ farad}$$
$$C_1 = 0.0002537 \text{ microfarad}$$

At 50,000 cycles,

$$\bar{y}_2 = \frac{1}{\bar{z}_2} = \frac{80}{(80)^2 + (3140)^2} - j\frac{3140}{(80)^2 + (3140)^2}$$
$$= 8.1 \times 10^{-6} - j318 \times 10^{-6} \text{ mho}$$
$$\bar{y}_1 = \frac{1}{\bar{z}_1} = 0 + j2\pi \times 5 \times 10^4 \times 0.2537 \times 10^{-9}$$
$$= 0 + j79.7 \times 10^{-6} \text{ mho}$$
$$\bar{y}_{12} = \bar{y}_1 + \bar{y}_2 = \text{admittance of 1 and 2 in parallel}$$
$$= 8.1 \times 10^{-6} - j238.3 \times 10^{-6} \text{ mho}$$
$$\bar{z}_{12} = \frac{1}{\bar{y}_{12}} = 142 + j4197 \text{ ohms}$$

For series resonance at 50,000 cycles for the entire circuit, the resultant reactance of the entire circuit must be zero. Therefore,

$$-\frac{1}{\omega C_3} = -4197 \text{ ohms}$$
$$C_3 = \frac{10^6}{4197 \times 2\pi \times 50{,}000}$$
$$= 0.0007585 \text{ microfarad}$$

The total impedance of the entire circuit at 50,000 cycles is

$$\begin{aligned} \bar{z}_0 &= (15 + j0) + (142 + j4197) + (0 - j4197) \\ &= 157 + j0 \text{ ohms} \\ z_0 &= 157 \text{ ohms} \end{aligned}$$

At 100,000 cycles,

$$\begin{aligned} \bar{y}_2 &= \frac{1}{\bar{z}_2} = \frac{80}{(80)^2 + (6280)^2} - j\frac{6280}{(80)^2 + (6280)^2} \\ &= 0.00000203 - j0.0001594 \text{ mho} \\ \bar{y}_1 &= j2\pi \times 10^5 \times 0.0002537 \times 10^{-6} \\ &= j0.0001594 \text{ mho} \\ \bar{y}_{12} &= \bar{y}_1 + \bar{y}_2 \\ &= 2.03 \times 10^{-6} - j0 \text{ mho} \\ \bar{z}_{12} &= \frac{1}{\bar{y}_{12}} = \frac{1}{2.03 \times 10^{-6} - j0} \\ &= 492{,}000 + j0 \text{ ohms} \\ \bar{z}_3 &= 0 - j\frac{1}{2\pi \times 10^5 \times 0.7585 \times 10^{-9}} \\ &= 0 - j2100 \text{ ohms} \\ \bar{z}_0 &= (15 + j0) + (492{,}000 + j0) + (0 - j2100) \\ &= 492{,}000 - j2100 \\ z_0 &= 492{,}000 \text{ ohms} \end{aligned}$$

The ratio of the impedances of the entire circuit at the two frequencies is

$$\begin{aligned} \frac{z_0(\text{at } 100{,}000 \text{ cycles})}{z_0(\text{at } 50{,}000 \text{ cycles})} &= \frac{492{,}000}{157} \\ &= 3130 \end{aligned}$$

If the interfering waves and the waves for which response is desired were of the same intensity, the current in the receiving circuit due to the interfering frequency would be only $\frac{1}{3130}$ of that caused by the waves of the frequency to be detected.

Effect of Change of Reactance and Resistance with Current, Reactance and Resistance Functions of Current.—In all the equations involving resistance or reactance or both, which have been considered thus far, both resistance and reactance have been

assumed constant. The equations are true only under these conditions.

When a sinusoidal voltage is impressed on a circuit of constant impedance, the resultant current under steady conditions is also sinusoidal. When, however, either the resistance or the reactance of the circuit is not constant but varies with the current, *i.e.*, is a function of the current, a sinusoidal voltage impressed on the circuit gives rise to a non-sinusoidal current.

The inductance of a circuit containing iron cannot be constant, since the flux is not proportional to the current. The flux per ampere is a variable. The reactance of such a circuit at any fixed frequency is a function of the current, except at very low flux densities, or when the magnetic circuit contains an air gap, when it may be approximately constant.

The resistance of a circuit, which has so small a heat capacity that the variation in the I^2r loss during a cycle produces an appreciable change in temperature, must vary with the current during each cycle, unless the coefficient of resistance variation with temperature is zero. If the temperature coefficient is positive like that of a metal, a sinusoidal voltage produces a current wave which is flatter than a sine wave. The current contains harmonics. In such a case the power factor of the circuit cannot be unity, even if the circuit contains no reactance, since the product of effective current and effective voltage cannot be equal to the true power absorbed.

The average power absorbed in heating by a circuit containing a resistance which is a function of the current is

$$P = \frac{1}{T}\int_0^T i^2 r\,dt$$

$$= \frac{1}{T}\int_0^T i^2 \mathrm{f}(i)\,dt \qquad (83)$$

This integral can be equal to $(I_{r.m.s})^2 r$ only when r is independent of the current strength. The apparent resistance of a circuit is always

$$r = \frac{P}{I^2} \qquad (84)$$

where P is the average power absorbed due to the resistance and I is the root-mean-square value of the current. (See Effective Resistance, page 254.)

The expressions

$$r = \frac{p}{i^2} \quad \text{and} \quad r = \frac{v}{i}$$

where r, p and i are instantaneous values of resistance, power and current and v is the instantaneous voltage drop due to resistance, are always equal.

In practice, the change in temperature of a conductor with current during a cycle is usually too small to produce appreciable effect. Low-candle-power, high-voltage incandescent lamps, however, especially those like the tungsten lamp with extremely fine filaments, do show an appreciable variation in resistance with current during a cycle at frequencies even as high as 60 cycles.

If a circuit contains iron, there may be a very appreciable change in its inductance with current during each cycle, especially when the iron is worked at high saturation. Owing to this variation, marked harmonics may be present in the current which are not present in the impressed voltage. With a sinusoidal voltage impressed on the circuit, a third harmonic with a maximum value as great as 25 to 40 per cent of the maximum value of the fundamental would not be excessive. A third harmonic of this magnitude often occurs in the no-load current of a transformer with sinusoidal impressed voltage.

The voltage drop across a circuit containing constant resistance and constant inductance is given by

$$e = ri + L\frac{di}{dt} \tag{85}$$

If $L = N\frac{d\varphi}{di}$ decreases with increase of current, as it must when iron is present, then, for a given impressed voltage, $\frac{di}{dt}$ must increase more rapidly than when L is constant in order that $L\frac{di}{dt} + ri$ shall be equal to the impressed voltage at each instant.

A sinusoidal voltage impressed on a circuit containing iron gives rise, therefore, to a current which is more peaked than a sinusoid.

Non-sinusoidal Voltage Impressed on a Circuit Containing Constant Impedances in Series and in Parallel.—When a non-sinusoidal voltage is impressed on a circuit consisting of constant resistances, constant inductances and constant capacitances in series or in parallel, the fundamental and each harmonic must be considered separately. They cannot be combined until the final solution is reached, when the root-mean-square value of the current and voltage drops may be found in the usual manner by taking the square root of the sum of the squares of the components caused by the fundamental and each harmonic acting separately. It must be remembered that reactance, susceptance and conductance are functions of frequency. For constant resistance, inductance and capacitance, inductive reactance is proportional to frequency and capacitive reactance is inversely proportional to frequency. Conductance decreases with increase in frequency for a circuit containing inductance and resistance only and increases with increase in frequency for a circuit containing capacitance and resistance only Conductance must be found for each frequency from the resistance, inductance and capacitance. Susceptance may either increase or decrease with frequency, depending on the relative magnitudes of resistance and reactance and the way the reactance changes with frequency. Susceptance, like conductance, must be found for each frequency from the resistance, inductance and capacitance.

Example of a Simple Series Circuit on Which a Non-sinusoidal Voltage Is Impressed.—A voltage whose equation is

$$e_0 \quad 300 \sin 314t + 75 \sin 942t$$

is impressed on a circuit consisting of a condenser of negligible resistance and 25 microfarads capacitance in series with an impedance of 40 ohms resistance and 0.1 henry inductance. What is the equation of the current and what is its root-mean-square value?

The impressed voltage contains a fundamental of

$$f = \frac{314}{2\pi} = 50 \text{ cycles}$$

and a harmonic of

$$f = \frac{942}{2\pi} = 150 \text{ cycles}$$

It therefore contains a fundamental and a third harmonic.
For the fundamental,

$$I_{m1} = \frac{E_{m1}}{\sqrt{r_0^2 + \left(2\pi f_1 L - \frac{1}{2\pi f_1 C}\right)^2}}$$
$$= \frac{300}{\sqrt{(40)^2 + \left(314 \times 0.1 - \frac{10^6}{314 \times 25}\right)^2}}$$
$$= \frac{300}{\sqrt{(40)^2 + (31.4 - 127.4)^2}} = \frac{300}{104.0}$$
$$= 2.884 \text{ amperes}$$
$$\tan \theta_1 = \frac{31.4 - 127.4}{40} = \frac{-96.0}{40} = -2.40$$
$$\theta_1 = -67.38 \text{ degrees}$$

For the third harmonic,

$$I_{m3} = \frac{E_{m3}}{\sqrt{r_0^2 + \left(2\pi f_3 L - \frac{1}{2\pi f_3 C}\right)^2}}$$
$$= \frac{75}{\sqrt{(40)^2 + \left(942 \times 0.1 - \frac{10^6}{942 \times 25}\right)^2}}$$
$$= \frac{75}{\sqrt{(40)^2 + (94.2 - 42.5)^2}} = \frac{75}{65.4}$$
$$= 1.146 \text{ amperes}$$
$$\tan \theta_3 = \frac{51.7}{40.0} = 1.293$$
$$\theta_3 = 52.3 \text{ degrees}$$
$$i_0 = 2.88 \sin (314t + 67.^\circ 4) + 1.15 \sin (942t - 52.^\circ 3)$$
$$I_{r.m.s} = \sqrt{\frac{(2.88)^2 + (1.15)^2}{2}}$$
$$= 2.19 \text{ amperes}$$

Effective Resistance.—When a direct current flows in a circuit, there is a loss in power due to the resistance of the circuit. This loss is the so-called *copper loss* and it is equal to the current squared, multiplied by the ohmic resistance of the circuit. The resistance is equal to the copper loss divided by the current squared. Similarly, when an equal alternating current flows in the same circuit there is also a loss in power, but, in general, the loss is greater with alternating current than with direct current. On account of this increase in loss, the apparent resistance of the circuit is greater with alternating current than with direct current, and, under certain conditions, it may be many times greater. The apparent resistance of a circuit with alternating current is always equal to the loss in power caused by the current divided by the current squared. The resistance of a circuit with direct current is not only equal to the loss caused by the current divided by the current squared, but it is also equal to the drop in potential across the circuit, caused by the current, divided by the current. The apparent resistance of a circuit with alternating current does not equal the drop in potential across it divided by the current, except in very special cases.

The increase in loss with alternating current is due to two causes, first, to certain local losses produced by the varying magnetic field in the surrounding material and in the conductor itself, and, second, to the non-uniform current density over the cross section of the conductor. With direct current, the current density is uniform over the cross section of ordinary conductors, *i.e.*, conductors of a single material of uniform specific resistance. The local losses produced by an alternating current are eddy-current and hysteresis losses in adjacent magnetic material, eddy-current losses in adjacent conducting material, and eddy-current loss and also hysteresis loss, if the conductor is magnetic, in the conductor itself.

When an alternating current flows in a circuit, the power absorbed is equal to the current multiplied by the active component of the voltage drop produced by the current. Any increase in the active component of the voltage drop, due to local losses produced by the current, is equivalent to an increase in the apparent resistance of the circuit. The active component of the voltage drop is increased by all local eddy-current and hysteresis

losses caused by the current and also by any distortion of the current distribution over the cross section of the conductor. The distortion of the current distribution has much the same effect, so far as the resistance of the conductor is concerned, as decreasing the cross section of the conductor. It thus increases its apparent resistance. The apparent resistance of a circuit to an alternating current is known as its *effective* resistance. Effective resistance is not a constant. It varies with frequency, and also with current strength, if the circuit is adjacent to magnetic material. In general, effective resistance increases rapidly with increase of frequency. In all work dealing with alternating currents, effective resistance must be used in finding the I^2r loss and the potential drop in the circuit due to resistance.

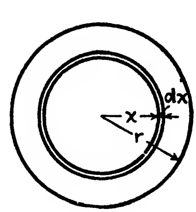


FIG. 70.

The non-uniform distribution of the current over the cross section of a conductor is due to the difference in the reactances of elements of the conductor taken parallel to its axis. Let Fig. 70 represent the cross section of a conductor. Consider any circular element in the conductor of radius x and width dx.

The flux lines caused by a current in a straight cylindrical conductor, which lies in a medium of uniform permeability, are concentric circles with their centers on the axis of the conductor. These flux lines not only surround the conductor but exist in the conductor itself. The field intensity at a point outside any cylindrical element, such as the one shown in Fig. 70, due to the current in the element, is equal to $\frac{2I}{R}\mu$, where I is the current in the element, R is the perpendicular distance between the point and the axis of the conductor and μ is the permeability of the medium within the element considered. The field intensity inside the element, due to the current it carries, is zero. Since the only flux which can link any element, such as the one of radius x shown in the figure, is that which lies without it, it is evident that the flux linking an element must increase as its radius decreases. All the flux produced by the entire conductor links the element at its center. This includes the flux within as well as without the conductor. The only flux that can link an element at the surface of the conductor is the flux which lies without the conductor.

The self-inductance per unit axial length of any element is equal to the flux linking it per unit length per unit current. This is equal to all that flux outside the element which is contained between two planes drawn perpendicular to the axis of the conductor at unit distance apart. It is obvious, therefore, that the self-inductance of the elements increases with decrease in their radii, is least for the element at the surface of the conductor and greatest for the element at the center. The reactance is, therefore, least for the element at the surface and greatest for the element at the center. Since reactance $2\pi fL$ is proportional to frequency, the difference between the reactance of an element at the center and at the surface of any conductor of fixed radius increases with frequency. It also increases with the radius of the conductor and its permeability. It is much greater for iron conductors than for those of copper.

The magnitude of the current in any element of a conductor varies inversely as the impedance of the element. The current, therefore, is greatest in elements at the surface and least in elements at the center. At very high frequencies, very little current flows through the central portion of a large conductor. Nearly all of the current is carried by the portion near its surface. This crowding of an alternating current toward the surface of a conductor is known as *skin effect.* It is small for conductors of small cross section carrying low-frequency currents, but for large conductors or high frequencies it may become very great. At very high frequencies, such as are used for radio communication, the resistance of a cylindrical tube may be very nearly as low as the resistance of a solid conductor of equal radius and the same material.

The skin effect can be calculated for non-magnetic conductors of circular section which are not adjacent to other conductors. For certain simple cross sections, other than circular, it is possible to calculate skin effect approximately. It can always be determined experimentally. When conductors are embedded in iron, as are the armature conductors of an alternator, it is also possible to calculate the skin effect with a fair degree of approximation.

The skin effect at 60 cycles for straight, stranded copper wires of circular cross section is shown in Fig. 71.

In general, the current density in a conductor is least at the points where the flux linkages per ampere due to the current in the conductor are greatest.

The apparent or effective resistance of a circuit not containing a source of electromotive force may be found by measuring the power absorbed when a known current of definite frequency is passed through it. If P is the power absorbed due to the current

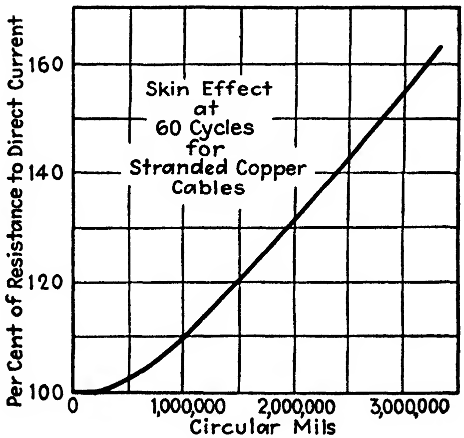


FIG. 71.

I at a frequency f, the effective resistance of the circuit at the frequency f is

$$r_e = \frac{P}{I^2} \tag{86}$$

Effective Reactance.—The effective reactance of a circuit to alternating current, especially if iron is present, is always less than the true reactance. The true reactance is equal to $2\pi fL$, where L is the self-inductance of the circuit and f is the frequency. The self-inductance L is equal to the flux linkages of the circuit *per unit current producing the flux.* The entire current in a circuit is not effective in producing flux except when no eddy-current or hysteresis losses are caused by it.

The power absorbed in any circuit, exclusive of that due to copper loss, is always equal to $EI \cos \theta_I^E$, where E and I are the back electromotive force and the current, respectively. The angle θ_I^E is the phase angle between E and I. If there is no

power absorbed by the circuit other than the true ohmic copper loss, the angle θ_I^E between E and I must be 90 degrees and E and I are in quadrature. Under this condition, $EI \cos \theta_I^E = 0$ and $VI \cos \theta_I^V = I^2 r_{oh}$ is the true copper loss, where V is the voltage drop impressed across the entire circuit and r_{oh} is the ohmic resistance of the circuit.

If there is magnetic or conducting material near the circuit, in which hysteresis or eddy-current losses occur, the current must have a component in phase opposition to the voltage *rise* induced by the flux, to supply these losses. This component does not contribute to the flux. Its effect, so far as flux is concerned, is canceled by the reaction due to the eddy-current and hysteresis losses. The only component of the current which is effective in producing flux is that in phase with the flux and, therefore, in quadrature with the voltage induced by the flux. This component is equal to the current that would be required to produce the flux if there were no eddy-current or hysteresis losses. It is called the *magnetizing component of the current* or simply the *magnetizing current.*

Consider a coil of wire wound on an iron core. If an alternating current is passed through the coil, a flux is set up which links the turns of the coil. Assume that this current is sinusoidal. Then, under the condition of constant permeability, the self-inductance is constant and the voltage drop due to self-inductance is sinusoidal. The current, as has been stated, is not in phase with this flux if there are eddy-current or hysteresis losses. Let L be the flux linkages with the circuit per unit current producing the flux, *i.e.*, per unit of the component of current in phase with the flux. Call this component $\bar{I}_\varphi$. Then the induced voltage drop through the coil, caused by its true self-inductance L, is $j2\pi f L \bar{I}_\varphi$.

Let $\bar{\varphi}$, Fig. 72, be the flux produced by the current $\bar{I}_\varphi$ and let $\bar{E} = j2\pi f L \bar{I}_\varphi$ be the voltage drop caused by this flux, where $\bar{I}_\varphi$ is the component of the current $\bar{I}$ in phase with the flux $\bar{\varphi}$. The flux and voltage drop must be in quadrature, with the voltage drop leading the flux. This phase relation follows from

$$e_{drop} = L\frac{di}{dt}$$

If the current is a sine function of time, the voltage drop e_{drop}, which is proportional to the derivative of the current with respect to time, must be a cosine function of time, and, since the cosine of an angle is equal to the sine of the angle plus 90 degrees, the drop must lead the current by 90 degrees.

The drop due to ohmic resistance is in phase with the current. Lay this off as shown. $\bar{V}$ is the total voltage drop across the circuit and is equal to the true self-induced voltage drop $j2\pi fL\bar{I}_\varphi$ plus the ohmic resistance drop $\bar{I}r$. The effective resistance is

$$r_e = \frac{P}{I^2} = \frac{VI\cos\theta}{I^2} = \frac{ac \text{ on Fig. 72}}{I}$$

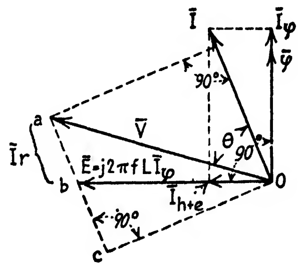


FIG. 72.

The true ohmic resistance drop is ab. The distance ac is the active component of the voltage drop with respect to the current. This is the apparent resistance drop. Similarly, $ob = 2\pi fLI_\varphi$ is the true reactance drop. The distance oc is the reactive component of the voltage drop with respect to the current. It is in quadrature with the total current $\bar{I}$ and is the apparent or effective reactance drop. The apparent or effective reactance, therefore, is equal to the reactive component of the voltage drop across the circuit divided by the total current. The true reactance is equal to the vector difference of the voltage drop across the circuit and the true ohmic resistance drop divided by the magnetizing component of the current, *i.e.*, by the component of the current in quadrature with the vector difference of the voltage drop across the circuit and the true ohmic resistance drop.

$$\text{True reactance } x = 2\pi fL = \frac{ob \text{ on Fig. 72}}{I_\varphi}$$

$$\text{Effective reactance } x_e = \frac{oc \text{ on Fig. 72}}{I}$$

$$\text{Effective impedance } z_e = \frac{V}{I} = \sqrt{r_e^2 + x_e^2}$$

The effective reactance is always less than the true reactance, and the effective resistance is always greater than the true resistance. The differences are greatest when the local losses

caused by the current are greatest. For circuits in which those losses are small, the effective reactance may be very nearly equal to the true reactance and the effective resistance may be very nearly equal to the true resistance.

In general, if a circuit containing resistance and reactance takes P watts and I amperes at V volts,

$$r_e \text{ (effective)} = \frac{P}{I^2} \text{ ohms} \tag{87}$$

$$z_e \text{ (effective)} = \frac{V}{I} \text{ ohms} \tag{88}$$

$$x_e \text{ (effective)} = \sqrt{z_e^2 - r_e^2} \text{ ohms} \tag{89}$$

$$g_e \text{ (effective)} = \frac{P}{V^2} \text{ mhos} \tag{90}$$

$$y_e \text{ (effective)} = \frac{I}{V} \text{ mhos} \tag{91}$$

$$b_e \text{ (effective)} = \sqrt{y_e^2 - g_e^2} \text{ mhos} \tag{92}$$

Equivalent Resistance, Reactance and Impedance of a Circuit and also Equivalent Conductance, Susceptance and Admittance of a Circuit.—It is frequently convenient, when dealing with certain problems, to replace an actual load, which may consist of a loaded motor or any other load, by an impedance or admittance which takes the same power at the same current and power factor as the actual load. So far as the conditions existing in the circuit are concerned, the impedance or admittance exactly replaces the actual load. Since the current, power and power factor of commercial loads, such as motor loads, are continually changing, the apparent impedance or admittance of a load cannot, in general, be constant.

The equivalent constants of any load may be found from equations (87) to (92), inclusive, by substituting for P, I and V the power, current and voltage of the load.

Example of the Use of the Equivalent Constants of a Load.—An induction motor and a synchronous motor, each rated at 2300 volts, are to be operated in parallel at the end of a transmission line having a resistance $r_L = 1.0$ ohm and an inductive reactance $x_L = 1.1$ ohms. The average power and current taken by each motor when operated at rated voltage with the particular load it has to carry are:

$$\text{Induction motor}\begin{cases} V_i = 2300 \text{ volts} \\ I_i = 111.5 \text{ amperes} \\ P_i = 200 \text{ kilowatts} \end{cases}$$

$$\text{Synchronous motor}\begin{cases} V_s = 2300 \text{ volts} \\ I_s = 100.0 \text{ amperes} \\ P_s = 150 \text{ kilowatts} \end{cases}$$

What must be the voltage at the power-station end of the line in order to maintain 2300 volts at the motors when they are operating under average load conditions? What is the resultant power factor of the load measured at the motors? What is the resultant power factor measured at the station end of the line? An induction motor always takes a lagging current. The excitation of the synchronous motor is adjusted so that it takes a leading current.

The equivalent constants of the motor loads are:

Induction Motor

$$y_i = \frac{111.5}{2300} = 0.0485 \text{ mho}$$

$$g_i = \frac{200 \times 1000}{(2300)^2} = 0.0378 \text{ mho}$$

$$b_i = \sqrt{y_i^2 - g_i^2}$$

$$= \sqrt{(0.0485)^2 - (0.0378)^2} = 0.0304 \text{ mho}$$

Synchronous Motor

$$y_s = \frac{100.0}{2300} = 0.0435 \text{ mho}$$

$$g_s = \frac{150 \times 1000}{(2300)^2} = 0.02835 \text{ mho}$$

$$b_s = \sqrt{y_s^2 - g_s^2}$$

$$= \sqrt{(0.0435)^2 - (0.02835)^2} = -0.03295 \text{ mho}$$

The equivalent susceptance b_s of the synchronous motor is negative since the synchronous motor takes a leading current.

$$g_0 = g_i + g_s = 0.0378 + 0.02835$$

$$= 0.06615 \text{ mho}$$

$$b_0 = b_i + b_s = 0.0304 - 0.03295$$

$$= -0.00255 \text{ mho}$$

$$r_0 = \frac{g_0}{g_0^2 + b_0^2} = \frac{0.06615}{(0.06615)^2 + (0.00255)^2} = \frac{0.06615}{0.004385}$$
$$= 15.08 \text{ ohms}$$

$$x_0 = \frac{b_0}{g_0^2 + b_0^2} = \frac{-0.00255}{(0.06615)^2 + (0.00255)^2} = \frac{-0.00255}{0.004382}$$
$$= -0.582 \text{ ohm}$$

$$y_0 = \sqrt{(0.06615)^2 + (0.00255)^2}$$
$$= 0.06625 \text{ mho}$$

The resultant power factor of the load, measured at the motors, is

$$\text{Power factor (at motors)} = \frac{g_0}{y_0} = \frac{0.06615}{0.06625} = 0.998$$

The current taken by the two motors in parallel is

$$I_0 = V_0 y_0$$
$$= 2300 \times 0.06625$$
$$= 152.4 \text{ amperes}$$

This current must lead the voltage V_0 impressed on the motors in parallel since $\tan \theta_0 = \frac{b_0}{g_0} = \frac{-0.00255}{0.06615}$ is negative.

The voltage V_0' at the station which gives 2300 volts at the motors is

$$V_0' = I_0\sqrt{(r_0 + r_L)^2 + (x_0 + x_L)^2}$$
$$= 152.4\sqrt{(15.08 + 1)^2 + (-0.582 + 1.1)^2}$$
$$= 152.4\sqrt{(16.08)^2 + (0.518)^2}$$
$$= 152.4 \times 16.09$$
$$= 2452 \text{ volts}$$

The power factor at the station is

$$\text{Power factor (at station)} = \frac{r_0 + r_L}{\sqrt{(r_0 + r_L)^2 + (x_0 + x_L)^2}}$$
$$= \frac{16.08}{16.09}$$
$$= 0.999$$

The current lags the voltage at the station, since $\tan \theta_{station} = \frac{x_0 + x_L}{r_0 + r_L}$ is positive.

General Summary of the Conditions in Series and Parallel Circuits.—Sinusoidal current and voltage waves are assumed.

Series

$\bar{I}_0 = \bar{I}_1 = \bar{I}_2 = \bar{I}_3 =$ etc.

$\bar{V}_0 = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 +$ etc.

$r_0 = r_1 + r_2 + r_3 +$ etc.

$x_0 = x_1 + x_2 + x_3 +$ etc.

$\bar{z}_0 = r_0 + jx_0$

$z_0 = \sqrt{r_0^2 + x_0^2}$

z's cannot be added directly.

Power factor $= \cos\theta_0 = \dfrac{r_0}{\sqrt{r_0^2 + x_0^2}}$

$\bar{V}_0 = \bar{I}_0\bar{z}_0 = \bar{I}_0(r_0 + jx_0)$ is an equation of voltage.

$\bar{I}_0 r_0 =$ active component of voltage drop with respect to current.

$j\bar{I}_0 x_0 =$ reactive component of voltage drop with respect to current.

Power $= P_0 = I_0^2 r_0$

$$\bar{z}_0 = \frac{\bar{V}_0}{\bar{I}_0} = r_0 + j\,x_0$$

$$z_0 = \frac{V_0}{I_0} = \sqrt{r_0^2 + x_0^2}$$

$$r_0 = \frac{P_0}{I_0^2}$$

$$x_0 = \sqrt{z_0^2 - r_0^2}$$

$$r_0 = \frac{g_0}{g_0^2 + b_0^2}$$

$$x_0 = \frac{b_0}{g_0^2 + b_0^2}$$

Parallel

$\bar{V}_0 = \bar{E}_1 = \bar{E}_2 = \bar{E}_3 =$ etc.

$\bar{I}_0 = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 +$ etc.

$g_0 = g_1 + g_2 + g_3 +$ etc.

$b_0 = b_1 + b_2 + b_3 +$ etc.

$\bar{y}_0 = g_0 - jb_0$

$y_0 = \sqrt{g_0^2 + b_0^2}$

y's cannot be added directly.

Power factor $= \cos\theta_0 = \dfrac{g_0}{\sqrt{g_0^2 + b_0^2}}$

$\bar{I}_0 = \bar{V}_0\bar{y}_0 = \bar{V}_0(g_0 - jb_0)$ is an equation of current.

$\bar{V}_0 g_0 =$ active component of current with respect to voltage drop.

$-j\bar{V}_0 b_0 =$ reactive component of current with respect to voltage drop.

Power $= P_0 = V_0^2 g_0$

$$\bar{y}_0 = \frac{\bar{I}_0}{\bar{V}_0} = g_0 - j\,b_0$$

$$y_0 = \frac{I_0}{V_0} = \sqrt{g_0^2 + b_0^2}$$

$$g_0 = \frac{P_0}{V_0^2}$$

$$b_0 = \sqrt{y_0^2 - g_0^2}$$

$$g_0 = \frac{r_0}{r_0^2 + x_0^2}$$

$$b_0 = \frac{x_0}{r_0^2 + x_0^2}$$

Series Resonance

For series resonance,

$$\Sigma x = x_0 = 0$$

The resultant impedance for fixed resistance is a minimum and for fixed impressed voltage the current is a maximum.

With fixed impressed voltage, the voltage drops across the inductance and capacitance may be excessive.

Parallel Resonance

For parallel resonance,

$$\Sigma b = b_0 = 0$$

The resultant admittance for fixed conductance is a minimum (the resultant impedance is a maximum) and for fixed impressed voltage the current is a minimum.

With fixed resultant current, the currents in the inductive and capacitive branches may be excessive.

If the resistance is low compared with the inductive and capacitive reactances, the voltage drops across the inductive and capacitive reactances may be dangerously large.

There can be no dangerously high voltages in any part of the circuit. If the resistances of parallel inductive and capacitive branches are very small, the currents in the branches may be very large.

For fixed current and fixed resultant resistance and therefore for fixed power ($P_0 = I_0^2r_0$), the resultant voltage drop across the entire circuit is a minimum. This is a very desirable condition for a constant-current power circuit.

For fixed voltage and fixed resultant conductance and therefore for fixed power ($P_0 = V_0^2g_0$), the resultant current in the circuit is a minimum. This is a very desirable condition for a constant-potential power circuit.

CHAPTER VIII

ELECTRIC FILTERS

Electric Filters.—In a broad sense, an electric filter is an electric circuit consisting of a number of impedances grouped together in such a way as to have definite frequency characteristics. The term in its more strict use and as employed in connection with communication circuits is applied to a network which consists of a single coupled circuit or a number of identical coupled circuits connected in series, which are designed to transmit freely over a certain range of frequencies and to transmit poorly over another range of frequencies. The range of frequencies over which free transmission occurs is called the *pass band.* The range of frequencies over which poor transmission occurs is called the *attenuation band.* The frequency at which attenuation starts to increase rapidly is called the *cut-off frequency.*

Uses of Filters.—Filters have many uses, a few of which are: to remove a commutation ripple from the voltage of a direct-current line or to eliminate an alternating-current ripple from the voltage of a direct-current circuit which receives power from a mercury-arc rectifier; to eliminate an undesirable harmonic from an alternating-current line; to suppress waves of certain frequencies and to pass waves of other frequencies in certain communication circuits. In the system of carrier-wave telephony, the voice-frequency currents are superposed on a carrier frequency current which is a current of higher frequency than any of the voice frequencies to be transmitted. Several carrier-currents with their superposed voice-frequency currents are transmitted over the same wires. Band-pass filters make it possible to separate the different carrier currents with their superposed voice frequencies at the receiving station.

Types of Filter Sections.—The most important filter sections are the symmetrical T and π networks. The ordinary power-transmission line may be considered to be made up of a number

of T or π networks. T and π networks are shown in Figs. 73 and 74, respectively.

A symmetrical T-connected section has two identical series impedances which are marked $\frac{1}{2}z_1$ in Fig. 73. A symmetrical π-connected section has two identical shunt impedances which are marked $2z_2$ in Fig. 74. The usual practice is to call the total series impedance of a filter z_1 and the total shunt impedance z_2. This practice makes each of the series impedances of a T-connected section $\frac{1}{2}z_1$ and each of the two shunt impedances of the π-connected section $2z_2$. A number of identical symmetrical sections connected in series forms a recurrent network which is commonly used for telephone filters. The T-connected network, which is an impedance-coupled circuit, was used in Chapter VI as the equivalent circuit of a two-winding transformer.

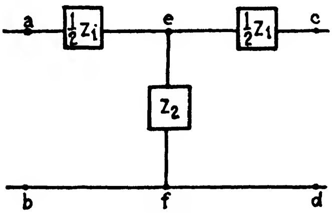


Fig. 73.

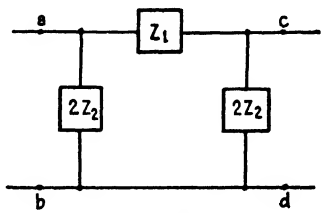


Fig. 74.

The lower the resistances of the impedances of a filter circuit and the greater the number of sections connected in series, the sharper is the cut-off between frequencies well and poorly transmitted.

The action of all filters depends upon the variation of inductive and capacitive reactances with frequency and upon the characteristics of series and parallel circuits. The frequency characteristic of any particular filter depends upon the kinds of impedances employed, *i.e.*, whether inductive or capacitive, and upon their grouping.

Types of Filters.—Filters are, in general, named in accordance with their frequency characteristics. There are four general types: low-pass filters, high-pass filters, band-pass filters and band-elimination filters. *Low-pass filters* transmit freely frequencies below a certain value and transmit poorly frequencies

above that value. *High-pass filters* transmit freely frequencies above a certain value and transmit poorly frequencies below that value. *Band-pass filters* transmit freely within a certain range of frequencies and transmit poorly on each side of that range. *Band-elimination filters* transmit poorly over a certain range of frequencies and transmit freely on each side of that range. A band-pass filter may be considered to be made up of a low-pass filter and a high-pass filter connected in series. To act as a band-pass filter, the two filters must have their regions of free transmission overlap. A low-pass filter and a high-pass filter in parallel act as a band-elimination filter, provided their regions of poor transmission overlap.

A Few Simple Types of Filters.—A condenser which is shunted across a direct-current line, fed from a mercury-arc rectifier, can be made to remove the ripple in the output voltage of the rectifier by virtually short-circuiting the ripple. A condenser has an infinite resistance to a direct current, if leakance is neglected, and a reactance which can be made low to the frequencies of the ripple by suitably adjusting its capacitance. A ripple of any definite frequency can be virtually short-circuited by a low-resistance shunt circuit, consisting of an inductive reactance and a capacitive reactance in series, which is tuned for series resonance for the frequency of the ripple to be removed.

A low-resistance inductive impedance, which is grounded at its mid-point and is connected between the conductors of a telephone line which is adjacent to a power line, can be used to filter off the electrostatic charge induced on the telephone line by the transmission line. The charges induced on the two telephone conductors are in phase, and the currents caused by them to ground in the two halves of the shunt impedance are also in phase when considered toward the grounded point. The reactance to ground is, therefore, very low, especially when the low frequency of the charges is considered. A current flowing through the impedance between the telephone conductors is the same in both halves of the impedance. Consequently, the impedance presents a large reactance to telephone frequencies and has very little short-circuiting effect on the telephone currents. Such a coil is usually spoken of as a *drainage* coil.

As they are not networks, the filters just described do not come under the strict definition of a filter as given in the first paragraph of this chapter.

Ladder Type of Filter.—One of the most important types of filter is the ladder type shown in Fig. 75. This consists of a

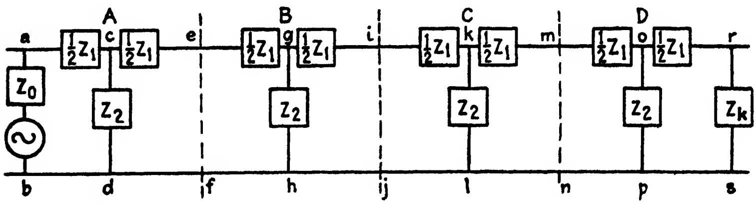


Fig. 75.

number of identical, symmetrical, T-connected sections connected in series. Four T sections are shown in Fig. 75. These are separated by dotted lines. Since the T sections are symmetrical, all of the $\frac{1}{2}z_1$'s in Fig. 75 are identical. All of the z_2's are also identical.

Another similar ladder filter, which is shown in Fig. 76, consists of a number of π sections in series. These sections are separated by dotted lines.

Characteristic Impedance of a T Section.—It is the symmetry of the T and π sections of the filters shown in Fig. 75 and Fig. 76 that gives them properties which are desirable for filter circuits.

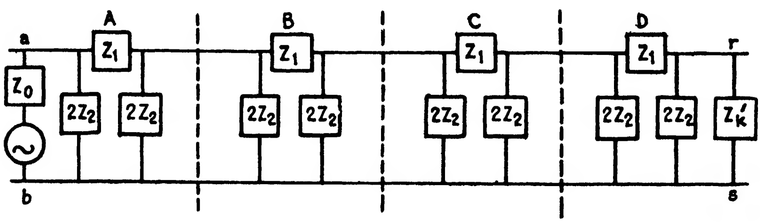


Fig. 76.

Refer to Fig. 75 and consider the right-hand section. For any particular values of the complex impedances $\frac{1}{2}\bar{z}_1$ and $\bar{z}_2$, there is a definite complex impedance $\bar{z}_K$ which, when connected across the output terminals r and s of the section D, makes the impedance at the input terminals m and n of the last section also $\bar{z}_K$.

The admittance $\bar{y}_{op}$ to the right of the points o and p is

$$\bar{y}_{op} = \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_K + \frac{\bar{z}_1}{2}}$$

$$\bar{z}_{op} = \frac{1}{\bar{y}_{op}} = \frac{\bar{z}_2(2\bar{z}_K + \bar{z}_1)}{2\bar{z}_K + \bar{z}_1 + 2\bar{z}_2} \tag{1}$$

The impedance between the input terminals of the section is, therefore,

$$\bar{z}_{mn} = \frac{\bar{z}_1}{2} + \bar{z}_{op} = \frac{\bar{z}_1}{2} + \frac{\bar{z}_2(2\bar{z}_K + \bar{z}_1)}{2\bar{z}_K + \bar{z}_1 + 2\bar{z}_2} \tag{2}$$

To find the value of $\bar{z}_K$ which makes the impedance across the input terminals of the section also $\bar{z}_K$, equate $\bar{z}_{mn}$ from equation (2) to $\bar{z}_K$.

$$\bar{z}_K = \frac{\bar{z}_1}{2} + \bar{z}_{op} = \frac{\bar{z}_1}{2} + \frac{\bar{z}_2(2\bar{z}_K + \bar{z}_1)}{2\bar{z}_K + \bar{z}_1 + 2\bar{z}_2}$$

from which

$$\bar{z}_K = \sqrt{\frac{(\bar{z}_1)^2}{4} + \bar{z}_1\bar{z}_2} \tag{3}$$

$$= \sqrt{\bar{z}_1\bar{z}_2\left(1 + \frac{\bar{z}_1}{4\bar{z}_2}\right)} \tag{4}$$

Similarly, it can be shown that for a π section (see Fig. 76),

$$\bar{z}_K' = \sqrt{\frac{\bar{z}_1\bar{z}_2}{1 + \frac{\bar{z}_1}{4\bar{z}_2}}} \tag{5}$$

where the z's have the significance indicated on Fig. 76.

The impedance $\bar{z}_K$, given by equation (4) for the T section, and the impedance $\bar{z}_K'$, given by equation (5) for the π section, are called the *characteristic impedances* of the sections. The impedance $\bar{z}_1$ is known as the *series-arm* impedance of the section The impedance $\bar{z}_2$ is known as the *shunt-arm* impedance.

Propagation, Attenuation and Phase Constants of the T-type Ladder Filter.—These same constants also apply to a power line, as such a line can be represented by a ladder-type structure

like that shown in Fig. 75. Let the terminal impedance of the filter be made equal to the characteristic impedance of the filter section as defined by equation (4) or (5). The impedance at any pair of junction points between the sections, such as the points i and j, e and f etc., Fig. 75, is also equal to $\bar{z}_K$. The fact that, when the terminal impedance is made equal to the characteristic impedance of the filter, this characteristic impedance reoccurs at each pair of junction points of the filter sections, gives the name *recurrent network* to the filter, and also the name *iterative impedance* to the characteristic impedance.

Let the capital letters A, B, C etc. designate the filter sections. Use these same letters as subscripts with the letter I to indicate the currents entering the T sections. Use the subscripts a, b, c etc. with the letter I to indicate the currents in the shunt branches of the sections. Let the positive directions of the currents be fixed as indicated below:

$$\bar{I}_{ac} = \bar{I}_A, \qquad \bar{I}_{cg} = \bar{I}_B \text{ etc.}$$
$$\bar{I}_{cd} = \bar{I}_a, \qquad \bar{I}_{gh} = \bar{I}_b \text{ etc.}$$

then,

$$\begin{aligned} \bar{I}_{ac} &= \bar{I}_{cd} + \bar{I}_{cg} \\ \bar{I}_A &= \bar{I}_a + \bar{I}_B \end{aligned} \tag{6}$$

Since the impedance between any two junction points between the sections is the characteristic impedance $\bar{z}_K$, the voltage drop between c and d is

$$\begin{aligned} \bar{V}_{cd} = \bar{I}_{cd}\bar{z}_2 &= \bar{I}_{cg}\left(\frac{\bar{z}_1}{2} + \bar{z}_K\right) \\ \bar{I}_a\bar{z}_2 &= \bar{I}_B\left(\frac{\bar{z}_1}{2} + \bar{z}_K\right) \\ \bar{I}_a &= \bar{I}_B\left(\frac{\frac{\bar{z}_1}{2} + \bar{z}_K}{\bar{z}_2}\right) \end{aligned} \tag{7}$$

Putting this value of $\bar{I}_a$ in equation (6) gives

$$\frac{\bar{I}_A}{\bar{I}_B} = \frac{\bar{z}_K + \frac{\bar{z}_1}{2} + \bar{z}_2}{\bar{z}_2} \tag{8}$$

In a similar way, it can be shown that

$$\frac{\bar{I}_B}{\bar{I}_C} = \frac{\bar{z}_K + \dfrac{\bar{z}_1}{2} + \bar{z}_2}{\bar{z}_2} \tag{9}$$

From the symmetry of the circuit, it follows that equation (9) gives the ratio of the vector current entering and leaving any section when the section is terminated in the characteristic impedance $\bar{z}_K$.

Let the letters A, B, C etc. by which the sections are designated be used as subscripts with the letter V to indicate the voltages across the input terminals of the sections. Let the positive directions of the voltages be as indicated below:

$$\bar{V}_{ab} = \bar{V}_A, \qquad \bar{V}_{ef} = \bar{V}_B \text{ etc.}$$

Since the impedance of the circuit between the input terminals of any section, such as the terminals e and f of the section B, is $\bar{z}_K$, it follows that the current entering any section is equal to the input voltage of the section divided by $\bar{z}_K$. Therefore,

$$\bar{I}_A = \frac{\bar{V}_A}{\bar{z}_K}, \qquad \bar{I}_B = \frac{\bar{V}_B}{\bar{z}_K} \text{ etc.} \tag{10}$$

Substituting in equation (8) the values of $\bar{I}_A$ and $\bar{I}_B$ in terms of the voltages $\bar{V}_A$ and $\bar{V}_B$, as given by equations (10),

$$\frac{\bar{V}_A}{\bar{V}_B} = \frac{\bar{z}_K + \dfrac{\bar{z}_1}{2} + \bar{z}_2}{\bar{z}_2} \tag{11}$$

The same expression holds for the ratio of the input and output voltages of any section. This ratio of the input and output voltages of any section is a constant complex number, which is equal to

$$\frac{\bar{z}_K + \dfrac{\bar{z}_1}{2} + \bar{z}_2}{\bar{z}_2} = a + jb = k\epsilon^{j\beta} \tag{12}$$

where $k = \sqrt{a^2 + b^2}$ and β is the angle determined by the relation

$$\beta = \tan^{-1} \frac{b}{a} \tag{13}$$

It follows from equation (11) that

$$\frac{\bar{V}_A}{\bar{V}_B} = \frac{\bar{I}_A}{\bar{I}_B} = k\epsilon^{j\beta} \tag{14}$$

$$\frac{\bar{V}_A}{\bar{V}_C} = \frac{\bar{I}_A}{\bar{I}_C} = \frac{\bar{V}_A}{\bar{V}_B} \times \frac{\bar{V}_B}{\bar{V}_C} = k^2\epsilon^{j2\beta} \tag{15}$$

$$\frac{\bar{V}_A}{\bar{V}_D} = \frac{\bar{I}_A}{\bar{I}_D} = k^3\epsilon^{j3\beta} \tag{16}$$

and for the nth section,

$$\frac{\bar{V}_A}{\bar{V}_N} = \frac{\bar{I}_A}{\bar{I}_N} = k^n\epsilon^{jn\beta} \tag{17}$$

The ratio of the current entering the first section to the currents in successive sections and also the ratio of the voltages across successive sections diminishes by the constant factor $k\epsilon^{j\beta}$.

$$\log_\epsilon (k\epsilon^{j\beta})^n = n\,(\log_\epsilon k + j\beta) \tag{18}$$

Calling

$$(\log_\epsilon k + j\beta) = \bar{\gamma}$$

equation (17) becomes

$$\frac{\bar{V}_A}{\bar{V}_N} = \frac{\bar{I}_A}{\bar{I}_N} = \epsilon^{n\bar{\gamma}} \tag{19}$$

Let

$$\log_\epsilon k = \alpha,$$

then

$$\bar{\gamma} = \alpha + j\beta$$

and

$$\frac{\bar{V}_A}{\bar{V}_N} = \frac{\bar{I}_A}{\bar{I}_N} = \epsilon^{n(\alpha+j\beta)} \tag{20}$$

The complex quantity $\bar{\gamma}$ in equation (19) is called the *propagation constant.* It determines both the change in magnitude and the change in phase of the current and the voltage in passing successive sections. The real part of $\bar{\gamma}$, *i.e.*, $\alpha = \log_\epsilon k$, is called the *attenuation constant.* It is the logarithm to the base ϵ of the change in the magnitude of the current and the voltage in passing successive sections. The angle β is called the *phase constant* and gives the increase in lag in phase of the current and the voltage in passing through successive sections.

From equation (20),

$$\bar{I}_N = \bar{I}_A \epsilon^{-n(\alpha+j\beta)} \tag{21}$$
$$\bar{V}_N = \bar{V}_A \epsilon^{-n(\alpha+j\beta)} \tag{22}$$

Equations (21) and (22) are used for calculating the currents and voltages for any section of a recurrent T network which is terminated in the characteristic impedance $\bar{z}_K$ when the characteristic impedance and the propagation constant are known.

Fundamental Equation of a T-type Filter Section.—Assume that the load impedance is equal to the characteristic impedance. Refer to the first section of the filter shown in Fig. 75, page 268.

$$\bar{V}_A = \bar{V}_B + \bar{I}_B \frac{\bar{z}_1}{2} + \bar{I}_A \frac{\bar{z}_1}{2} \tag{23}$$

Substituting in equation (23) the values of $\bar{I}_A$ and $\bar{I}_B$ in terms of $\bar{V}_A$ and $\bar{V}_B$ from equation (10) gives

$$\frac{\bar{V}_A}{\bar{V}_B} = \frac{2\bar{z}_K + \bar{z}_1}{2\bar{z}_K - \bar{z}_1} \tag{24}$$

From equation (14),

$$\frac{\bar{V}_A}{\bar{V}_B} = \frac{2\bar{z}_K + \bar{z}_1}{2\bar{z}_K - \bar{z}_1} = k\epsilon^{j\beta} = \epsilon^{\bar{\gamma}} \tag{25}$$

Solving equation (25) for $\bar{z}_1$,

$$\bar{z}_1 = 2\bar{z}_K \frac{\epsilon^{\bar{\gamma}} - 1}{\epsilon^{\bar{\gamma}} + 1} \tag{26}$$

From equation (3), page 269,

$$\bar{z}_2 = \frac{4\bar{z}_K^2 - \bar{z}_1^2}{4\bar{z}_1} \tag{27}$$

Replacing $\bar{z}_1$ by its value in equation (26),

$$\bar{z}_2 = \frac{2\bar{z}_K \epsilon^{\bar{\gamma}}}{(\epsilon^{\bar{\gamma}} + 1)(\epsilon^{\bar{\gamma}} - 1)} \tag{28}$$

From equations (26) and (28),

$$\frac{\bar{z}_1}{\bar{z}_2} = 2\bar{z}_K \frac{\epsilon^{\bar{\gamma}} - 1}{\epsilon^{\bar{\gamma}} + 1} \times \frac{(\epsilon^{\bar{\gamma}} + 1)(\epsilon^{\bar{\gamma}} - 1)}{2\bar{z}_K \epsilon^{\bar{\gamma}}} \tag{29}$$

$$= \frac{\epsilon^{2\bar{\gamma}} - 2\epsilon^{\bar{\gamma}} + 1}{\epsilon^{\bar{\gamma}}}$$

$$= \epsilon^{\bar{\gamma}} + \epsilon^{-\bar{\gamma}} - 2 = 2 \cosh \bar{\gamma} - 2 \tag{30}$$

Substituting in equation (30) the value of $\bar{\gamma}$ in terms of the attenuation constant α and the phase constant β gives

$$\frac{\bar{z}_1}{\bar{z}_2} = \epsilon^{(\alpha+j\beta)} + \epsilon^{-(\alpha+j\beta)} - 2 = 2 \cosh (\alpha + j\beta) - 2 \tag{31}$$

$$1 + \frac{\bar{z}_1}{2\bar{z}_2} = \cosh (\alpha + j\beta) = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta \tag{32}$$

Since $\epsilon^{j\beta} = (\cos \beta + j \sin \beta)$ and ϵ^{-jb} $(\cos \beta - j \sin \beta)$ [see equations (32) and (33) page 17], equation (30) can also be written

$$\begin{aligned} 1 + \frac{\bar{z}_1}{2\bar{z}_2} &= \frac{\epsilon^{\alpha}}{2}(\cos \beta + j \sin \beta) + \frac{\epsilon^{-\alpha}}{2}(\cos \beta - j \sin \beta) \\ &= \frac{\epsilon^{\alpha} + \epsilon^{-\alpha}}{2} \cos \beta + j \frac{\epsilon^{\alpha} - \epsilon^{-\alpha}}{2} \sin \beta \\ &= \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta \end{aligned} \tag{33}$$

Equation (32) is the fundamental equation for the T-type filter section.

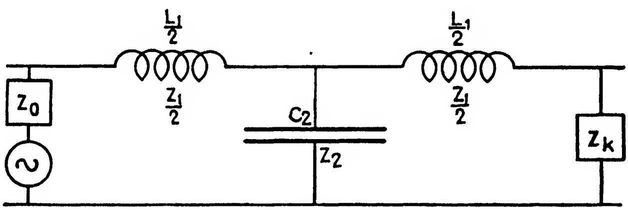


FIG. 77.

Fundamental Equation for the π-type Filter Section.—Equation (33) just developed for the T-type filter section can also be shown to hold for the π-type filter section.

Low-pass Filter.—A low-pass filter was defined on page 266. An ideal filter is one without resistances. This condition cannot be attained in practice, but it can be approached. In what follows an ideal filter is assumed, *i.e.*, one without resistances in its impedances. A simple, low-pass, T-connected filter section is shown in Fig. 77.

That the filter shown in Fig. 77 is a low-pass filter is obvious from its arrangement. The impedance in the series arm of the

filter is inductive and its reactance increases with frequency. On the other hand, the shunt-arm impedance is capacitive and its reactance decreases with increase in frequency. Therefore, its effect in shunting the current entering the filter increases with frequency. Although it can be seen that the action of the filter shown in Fig. 77 in preventing the passage of a current is greater at high frequencies than at low frequencies, its exact action cannot be seen without the use of the fundamental ladder-network equation which is given in equation (32), page 274.

Call the inductance of the series arm of the filter L_1. This corresponds to the inductance of both of the impedances $\frac{z_1}{2}$ in series. Let the capacitance of the shunt arm be C_2. Then, since the resistances are assumed to be zero, $z_1 = 2\frac{z_1}{2} = \omega L_1$ and $z_2 = \frac{-1}{\omega C_2}$.

$$1 + \frac{z_1}{2z_2} = 1 - \frac{\omega^2 L_1 C_2}{2} \tag{34}$$

Putting this in equation (33) gives

$$\begin{aligned} 1 - \frac{\omega^2 L_1 C_2}{2} &= \frac{\epsilon^\alpha + \epsilon^{-\alpha}}{2} \cos \beta + j\frac{\epsilon^\alpha - \epsilon^{-\alpha}}{2} \sin \beta \\ &= \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta \end{aligned} \tag{35}$$

If the filter is to pass waves without attenuation, the attenuation constant α must be zero. Putting $\alpha = 0$, equation (35) becomes

$$1 - \frac{\omega^2 L_1 C_2}{2} = \cos \beta \tag{36}$$

$\cos \beta$ can have any value between $+1$ and -1. When $\beta = 0$, $\cos \beta = 1$ and $\frac{\omega^2 L_1 C_2}{2}$ is zero. Since neither L_1 nor C_2 can be zero, ω must be zero. This limiting case corresponds to a direct current. A direct current would obviously pass without attenuation through the filter assumed.

When $\cos \beta$ has its other limiting value, *i.e.*, -1, equation (35) gives

$$1 - \frac{\omega^2 L_1 C_2}{2} = -1$$

$$\omega^2 L_1 C_2 = 4$$

$$\omega = \omega_c = \frac{2}{\sqrt{L_1 C_2}}$$

$$f_c = \frac{1}{\pi\sqrt{L_1 C_2}} \text{ cycles per second} \tag{37}$$

Equation (37) gives the cut-off frequency for the filter. Currents with frequencies from zero to f_c are therefore transmitted freely. For any frequency higher than the cut-off frequency f_c, given by equation (37), cos β, as given by equation (36), is greater than unity and β is imaginary if α is zero. Therefore, for frequencies above the cut-off frequency, attenuation and hence the attenuation constant cannot be zero.

Although currents with frequencies between zero and the cut-off frequency pass without attenuation, they are shifted in phase by an angle given by equation (36). At zero frequency, this angle is zero, and at the cut-off frequency the angle is 180 degrees. Hence, while there is no attenuation in an ideal filter between these frequencies, *i.e.*, within the pass band, there is a shift in phase which increases from zero to 180 degrees. At and above the cut-off frequency, there is a complete reversal in phase. The preceding statements assume that the output impedance is equal to the characteristic impedance for each frequency.

When the resistances of the arms of the filter are zero, equation (35) has no j or imaginary terms. This follows from equation (34), which can have no j term when the resistances of the arms of the filter are zero. Under this condition, equation (35) becomes

$$1 - \frac{\omega^2 L_1 C_2}{2} = \frac{1}{2}(\epsilon^{\alpha} + \epsilon^{-\alpha}) \cos \beta \tag{38}$$

For all values of frequency between $f = 0$ and $f = f_c$, the cut-off frequency, the attenuation constant α in equation (38) is zero. Above the cut-off frequency, the attenuation constant α can be found by giving to f_c values which are greater than the cut-off frequency. Above cut-off, there is a complete reversal in phase for each section of the filter.

The characteristic impedance of the filter from equation (3), page 269, is

$$\bar{z}_K = \sqrt{\frac{\bar{z}_1^2}{4} + \bar{z}_1\bar{z}_2}$$

$$= \sqrt{\frac{1}{4}(j\omega L_1)^2 + (j\omega L_1)\left(j\frac{-1}{\omega C_2}\right)}$$

$$= \sqrt{\frac{L_1}{C_2}}\sqrt{1 - \frac{1}{4}\omega^2 L_1 C_2} = \sqrt{\frac{L_1}{C_2}}\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \qquad (39)$$

This is not independent of frequency. Therefore, in order that there may be no attenuation of a current in passing through the filter, the load impedance must be equal to the characteristic impedance for the frequency of the current. For frequencies beyond the cut-off point, there is attenuation. The limiting values of the characteristic impedance in the pass band can be found by putting $f = 0$ and $f = f_c$, the cut-off frequency, in equation (39). These limiting values are $\sqrt{\frac{L_1}{C_2}}$ and 0.

Below cut-off frequency, $\left(\frac{f}{f_c}\right)^2$ is less than unity. Under this condition $\bar{z}_K$ is real and must be equivalent to a resistance. Above the cut-off frequency, $\left(\frac{f}{f_c}\right)^2$ is greater than unity. In this case $\sqrt{1 - \left(\frac{f}{f_c}\right)^2}$ reduces to a real term multiplied by $\sqrt{-1} = j$. Under this condition $\bar{z}_K$ must be equivalent to a reactance. It is inductive reactance, since the low-pass filter has series inductance and shunt capacitance and must therefore act as inductance at frequencies above those which make it act like a resistance, *i.e.*, above the cut-off frequency.

Typical-attenuation, Phase-shift and Characteristic-impedance Curves of Resistanceless Filter Sections.—Typical-attenuation, phase-shift and characteristic-impedance curves plotted against frequency are given in Fig. 78 for a resistanceless, single-section, low-pass, T-type filter like that shown in Fig. 77.

Similar curves are given in Fig. 80 for a resistanceless, single-section, low-pass, π-type filter like that shown in Fig. 79.

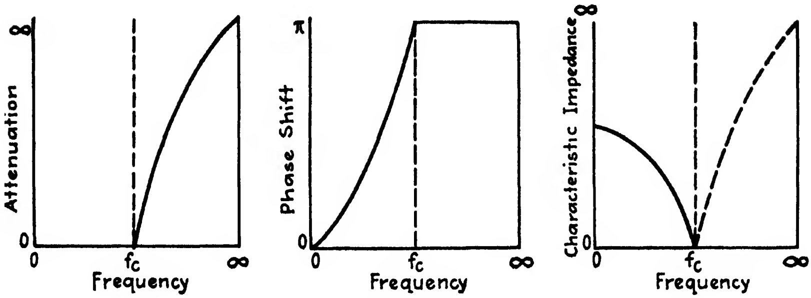


FIG. 78.

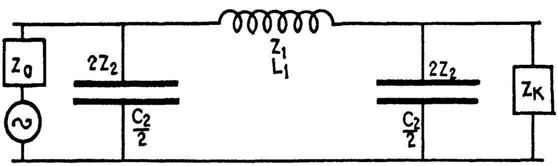


FIG. 79.

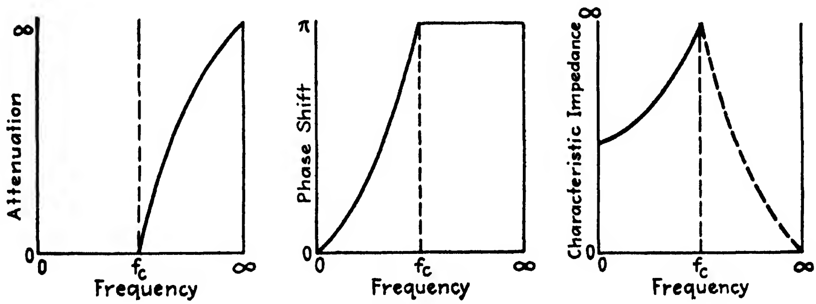


FIG. 80.

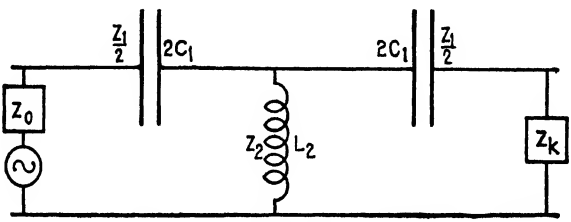


FIG. 81.

The characteristic impedance for the π-type, low-pass filter shown in Fig. 79 is

$$\bar{z}_K' = \sqrt{\frac{L_1}{C_2}} \div \sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{40}$$

The attenuation, the phase characteristic and the cut-off frequency, assuming no resistance in the filter, are the same as for T-type, low-pass filter.

High-pass Filter.—A single-section, high-pass, T-type filter is shown in Fig. 81.

The series-arm impedance for this filter is low for high frequencies because the series impedance is capacitive and decreases with an increase in frequency. The shunt impedance is inductive and its shunting effect decreases with an increase in frequency. It is obvious that this filter must tend to pass currents of high frequency more freely than currents of low frequency.

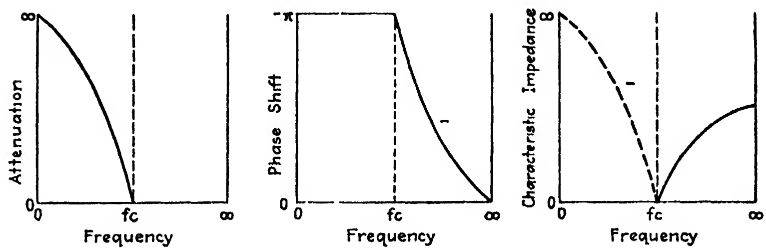


FIG. 82.

The equation for the characteristic impedance of this type of filter is

$$\bar{z}_K = \sqrt{\frac{L_2}{C_1}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{41}$$

The cut-off frequency is

$$f_c = \frac{1}{4\pi\sqrt{L_2C_1}} \tag{42}$$

Typical-attenuation, phase-shift and characteristic-impedance curves plotted against frequency are given in Fig. 82 for a resistanceless, single-section, high-pass filter like that shown in Fig. 81.

Similar curves are given in Fig. 84 for a resistanceless, single-section, π-type, high-pass filter like that shown in Fig. 83.

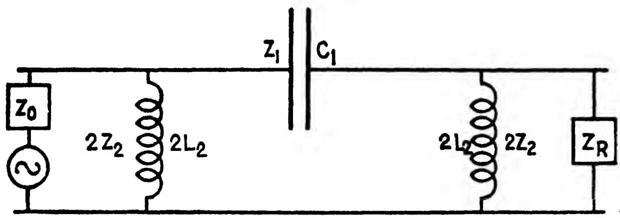


FIG. 83.

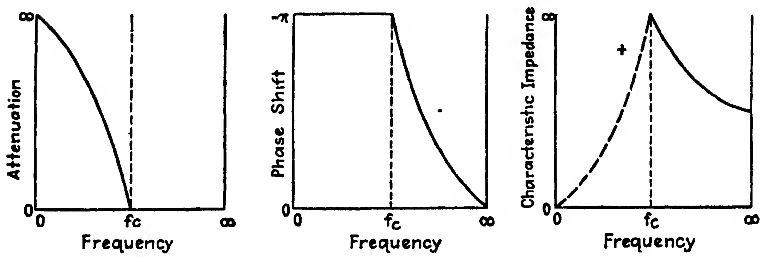


FIG. 84.

The characteristic impedance for the single-section, high-pass, π-type filter, shown in Fig. 83, is

$$\bar{z}_K' = \sqrt{\frac{L_2}{C_1}} \div \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{43}$$

Its cut-off frequency is

$$f_c = \frac{1}{4\pi\sqrt{L_2C_1}} \tag{44}$$

The complete theory of filter circuits is too extensive to consider in an elementary text on alternating currents. For further study the reader is referred to books on the subject.[1]

[1] Transmission Circuits for Telephonic Communication, K. S. Johnson. Transmission Networks and Wave Filters, T E. Shea.

CHAPTER IX

POLYPHASE VOLTAGES AND CURRENTS

Generation of Polyphase Voltages.—A polyphase alternator differs from a single-phase alternator only in the number of its armature windings. A single-phase alternator has a single armature winding. A polyphase alternator has as many independent armature windings as there are phases. These windings are displaced from one another by equal angles, the angles being determined by the number of phases. If a two-pole alternator has two independent armature windings displaced 90 degrees from each other, the coil sides of one winding lie under the centers of the poles when the coil sides of the other winding are midway between the poles. The voltages induced in the two windings, therefore, are in time quadrature or 90 degrees apart in time phase. If the terminals of the windings are brought out to insulated slip rings mounted on the armature shaft, two-phase currents can be taken from the alternator. If the alternator has three independent armature windings, displaced 120 degrees from one another, the voltages induced in them are 120 degrees apart in time phase. Three-phase voltages are generated and the alternator is a three-phase alternator. In general, the time-phase angle between the voltages of a polyphase alternator is equal to 360 degrees divided by the number of phases. This statement does not hold for a so-called "two-phase" alternator, which has the equivalent of two armature windings 90 degrees apart, instead of four as in the case of a four-phase machine. The only difference between two-phase and four-phase alternators is in the manner in which their armature windings are interconnected. A two-phase system is really half of a four-phase system.

The arrangement of a two-pole, two-phase alternator with a revolving armature is shown in Fig. 85. The slip rings for the collection of the two-phase currents are omitted to avoid confusion.

A polyphase alternator always has as many independent armature windings as there are phases. These are displaced from one another on the armature by 360 electrical degrees divided by the number of phases, except for two-phase. In space degrees this displacement corresponds to 360 degrees divided by the number of phases and by the number of pairs of poles.

With few exceptions, all modern alternators are three-phase. Very few two-phase or four-phase alternators are built, except for use in existing two-phase or four-phase plants. All power transmission is three-phase. With fixed voltage between any two conductors of a transmission line and fixed transmission loss, a greater amount of power can be transmitted a given distance three-phase than with any other number of phases. With fixed voltage to neutral, the amount of copper required for a transmission line is independent of the number of phases. The limiting condition, however, is not the voltage to neutral, but the maximum voltage between any pair of conductors. The cost of the greater number of insulators required for a four-phase transmission line over the number necessary for a three-phase transmission line would alone be a serious item.

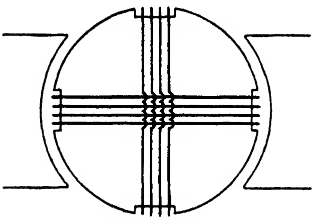


Fig. 85.

All commercial alternators have stationary armatures and rotating fields. Such an arrangement requires no slip rings for the collection of the armature current. Two slip rings are required for the field winding, but these have to carry only the low-voltage direct current necessary for excitation. The armature of an alternator consists of a laminated steel core, slotted on its inside for the armature winding. The poles revolve inside the armature core. This arrangement puts the more complicated winding, and the one which is subjected to high voltages, on the stationary part of the machine where it can be easily insulated. It also relieves the armature winding from all stresses except those due to the load.

A four-pole, three-phase alternator with a revolving field and a stationary armature is shown in Fig. 86, where 1-1′, 2-2′

and 3-3′ are the coils for the three phases. Since there are four poles, there are two groups of coils for each phase. The two groups of coils for any phase are 360 electrical degrees apart. The voltages induced in them, therefore, are in time phase. Since the voltages induced in the two coils for any phase are in time phase, these coils can be connected either in series or in parallel. They are, however, connected in series in most cases.

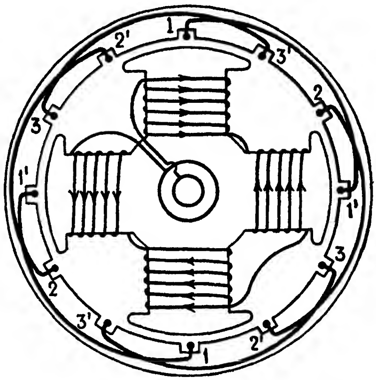


FIG. 86

All alternators, except those built to be driven by high-speed steam turbines, have more than two poles. Large low-speed alternators, such as those designed to be driven by low-head water wheels, may have as many as forty or fifty poles. The frequency of an alternator is independent of the number of phases for which it is wound. It depends only on the number of poles and the speed. Frequency is always given by

$$f = \frac{\text{rev. per min.}}{60} \times \text{no. pairs of poles} = \text{cycles per second} \quad (1)$$

The armature winding of each phase of multipolar alternators generally consists of as many identical groups of windings as there are pairs of poles. These are displaced 360 electrical degrees from one another on the armature and, therefore, have voltages induced in them which are in time phase. Since the voltages are in time phase, the groups of windings for any phase can be connected either in series or in parallel. Series connection, however, is the more common, as high voltage is usually desired. In general, the windings of a polyphase alternator are spread out so as to cover the entire armature periphery. This distributes the heating due to the armature copper loss and also improves the wave form. It decreases the resultant voltage obtained from a given number of armature turns and a fixed field flux, but this diminution is small in the case of polyphase alternators, only four or five per

cent, and the better distribution of the armature copper loss and the better wave form more than compensate for it.

The chief advantages of three-phase alternators over single-phase alternators are as follows:

1. They give about fifty per cent greater output for a given amount of material. For a given capacity, therefore, they are cheaper.

2. They are more satisfactory for the operation of motors, since polyphase motors have better operating characteristics than single-phase motors. Polyphase motors give the same increase in output for a given amount of material as polyphase alternators.

3. They are more efficient.

4. They are more satisfactory for power transmission, since with a fixed voltage between conductors, a fixed amount of power transmitted a given distance and a fixed transmission loss, twenty-five per cent less copper is required for three-phase transmission than for single-phase transmission. The copper efficiencies of single-phase and of four-phase transmission are the same.

Double Subscript Notation for Polyphase Circuits.—In working with current and voltage relations in the simplest polyphase circuits, the particular notation employed is not important. In the more complex cases, however, any lack of definiteness in the notation or in its interpretation is almost sure to result in confusion if not in serious error. A simple and satisfactory notation is based on lettering every junction and terminal point of diagrams of connections and on the use of two subscripts with every vector representing current or voltage.

The vector diagram must be distinguished from the diagram of connections, although in certain cases there may be some similarity between them. The subscripts, taken from the diagram of connections, indicate that the positive direction of the current is from the first to the second and that the positive direction of the electromotive force is also from the first to the second. The current $\bar{I}_{ab}$, according to this notation, is the current whose positive direction is from a to b in the branch ab of the circuit, and $\bar{E}_{ab}$ is the electromotive force which produces this current. Also, $\bar{I}_{ba}$ is the current whose positive direction is from b to a

and it is produced by the electromotive force $\bar{E}_{ba}$. Again, $\bar{I}_{ab}$ is equal to $-\bar{I}_{ba}$, and $\bar{I}_{ab}$ and $\bar{I}_{ba}$ differ in phase by 180 degrees. $\bar{E}_{ab}$ and $\bar{E}_{ba}$ also differ by 180 degrees. If the rotating vector marked $\bar{I}$, Fig. 21, page 71, is the current $\bar{I}_{ab}$, it is positive during the time its projection on the vertical axis is positive, *i.e.*, when the revolving vector $\bar{I}$ lies in the first or second quadrant. During this time, the current actually flows from *a* to *b* in the circuit *ab*. While the current vector $\bar{I}$ lies in the third or fourth quadrant, the current actually flows from *b* to *a*. If the vector $\bar{E}$, Fig. 21, page 71, represents the rise in potential $\bar{E}_{ab}$, there is an actual rise in potential in the direction *a* to *b* during the time the projection of the vector $\bar{E}$ on the vertical axis is positive. There is a fall of potential in the direction *ab* when the projection of the vector $\bar{E}$ on the vertical axis is negative. (See pages 68, 69, and 70.)

If $\bar{E}_{ab}$ represents a rise in potential, then, on the average, power is delivered during each cycle when $\bar{E}_{ab}$ and $\bar{I}_{ab}$ have components in phase, and power is absorbed when they have components in opposition. Power delivered is positive and power absorbed is negative. If $\bar{E}_{ab}$ represents a fall in potential, $-\bar{E}_{ab} = \bar{E}_{ba}$ is the corresponding potential rise. In this case, power is absorbed when $\bar{E}_{ab}$ and $\bar{I}_{ab}$ have components in phase, and it is delivered when they have components in opposition. If the circuit *ab* contains pure resistance, the current $\bar{I}_{ab}$ is in phase with the voltage drop produced by the current in the resistance. It is in phase with $\bar{E}_{ab}$ when the vector $\bar{E}_{ab}$ represents a voltage drop. It is in phase with $-\bar{E}_{ab}$ when the vector $\bar{E}_{ab}$ represents a voltage rise. If the circuit *ab* contains inductive reactance in addition to resistance, $\bar{I}_{ab}$ lags the voltage drop by an angle whose cosine is the ratio of the resistance to the impedance, *i.e.*, by the angle

$$\theta = \cos^{-1}\frac{r}{\sqrt{r^2 + x^2}} \tag{2}$$

where r and x are the resistance and reactance, respectively, of the circuit *ab*.

The current $\bar{I}_{ab}$ must always be used with the voltage $\bar{E}_{ab}$. The power in the circuit is $E_{ab}I_{ab}\cos\theta$ and never $E_{ab}I_{ba}\cos\theta$. If $\bar{E}_{ab}$ is a voltage rise, $E_{ab}I_{ab}\cos\theta$ is power generated. If $\bar{E}_{ab}$

is a voltage drop, $E_{ab}I_{ab} \cos \theta$ is power absorbed. If $E_{ab}I_{ab} \cos \theta$, representing power generated, is negative, it actually represents power absorbed, since negative power generated is power absorbed. If $E_{ab}I_{ab} \cos \theta$, representing power absorbed, is negative, it actually represents power delivered or generated.

The voltage drop E_{ab} due to a current in an impedance is

$$E_{ab} = I_{ab}\sqrt{r^2 + x^2}$$
$$\bar{E}_{ab} = \bar{I}_{ab}(r + jx)$$

It is *never* $E_{ab} = I_{ba}\sqrt{r^2 + x^2}$.

It is often necessary to use double subscripts with resistance, reactance and impedance and also with conductance, susceptance and admittance, but in such cases the subscripts have no other significance than to indicate between what two points of a circuit the quantities mentioned are measured. The order of the subscripts can mean nothing, since resistance, reactance, impedance etc. are not vectors.

When in specific problems exact numerical results are required, an analytical solution by the method of complex quantities is necessary. However, experience has shown conclusively that, even if a particular problem is to be solved analytically, an approximate vector diagram should always be drawn, as it facilitates the correct interpretation of the work and serves as a check against errors. It is usually advisable to draw all polyphase vectors radially from a common center. When vectors are drawn radially from a common center, there can be no question as to their phase relations, and, if each vector is lettered with two subscripts, there is no excuse for mistaking the angle between any two vectors, such as $\bar{E}_{ab}$ and $\bar{I}_{cd}$, for the angle between $\bar{E}_{ab}$ and $\bar{I}_{dc}$. Neither is there any excuse for mistaking the angle between $\bar{E}_{ab}$ and $\bar{I}_{ba}$ for the angle between $\bar{E}_{ab}$ and $\bar{I}_{ab}$. The one angle is the supplement of the other. Double subscripts will, in general, be used in what follows relating to polyphase circuits.

Wye and Delta Connections for Three-phase Generators and for Three-phase Circuits.—The windings of a three-phase alternator may be represented diagrammatically by three windings displaced 120 degrees from one another, as shown in Fig. 87.

The voltages generated in them are equal and differ 120 degrees in time phase. If the windings are assumed to rotate in a counter-clockwise direction, the voltage generated in the phase marked 0-1 leads the voltage generated in phase 0-2 by 120 degrees. It leads the voltage generated in phase 0-3 by 240 degrees. Three-phase voltages are generated in the windings. These may be represented by three vectors, $\bar{E}_{01}$, $\bar{E}_{02}$ and $\bar{E}_{03}$, which are equal in magnitude and 120 degrees apart in time phase. These are shown in Fig. 88. The order of the subscripts used with the vectors indicates the direction in which the voltages

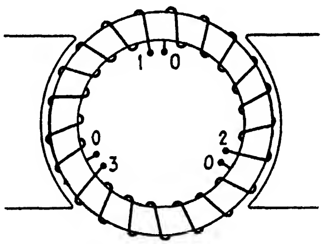


FIG. 87.

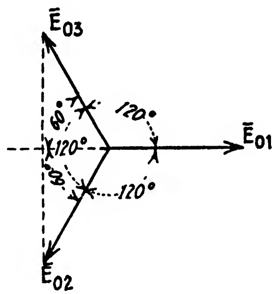


FIG. 88.

are considered with respect to the windings. All voltages must be considered in the same direction around the armature if they are to differ by 120 degrees in time phase.

The voltage vector $\bar{E}_{01}$ is taken along the axis of reals. The symbolic expressions for the three equal voltages are

$$\bar{E}_{01} = (1 - j0)E \tag{3}$$

$$\bar{E}_{02} = \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) E \tag{4}$$

$$\bar{E}_{03} = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) E \tag{5}$$

where E is the magnitude of the voltage generated in any one phase. From equations (3), (4) and (5)

$$\bar{E}_{01} + \bar{E}_{02} + \bar{E}_{03} = 0 \tag{6}$$

Since the vector sum of the voltages generated in the three windings is equal to zero when considered in the same direction around the armature, the terminals of windings 0-1, 0-2 and 0-3

may be connected to form a closed circuit, as shown in the left-hand half of Fig. 89, and no current then flows, since the resultant voltage acting in the closed circuit formed by the armature windings is zero.[1] The terminals of the alternator are taken from the junction points between the phases, *i.e.*, from the points 1, 2 and 3. The windings connected in this manner form a closed delta. This connection is known as the *delta* connection.

Instead of connecting the phases in delta, they may be connected to form a wye, by joining the corresponding terminals of the windings of the three phases. Either the terminals marked 0 or those marked 1, 2 and 3 may be joined for *wye* connection. Delta and wye connections, usually written Δ and Y, are illustrated in Fig. 89. The field poles are omitted in the figure.

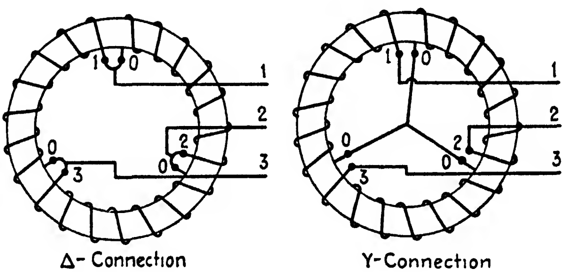


FIG. 89.

A Δ-connected alternator can have only three terminals. A Y-connected alternator need have only three, but in some cases a fourth is brought out from the common junction between the windings or *neutral point*, as this junction is called, to permit grounding the alternator. When loads are connected in wye, the neutral connection is usually employed. When the neutral point is available, the load may be applied between any two of the three line terminals, *i.e.*, between 1 and 2, between 2 and 3 or between 3 and 1, or it may be applied between any one of the line terminals and the neutral point. In the first case, the load is Δ-connected. In the second case, it is Y-connected. When the windings of an alternator are connected in delta, the load is usually connected in delta, although it may be connected in wye; but when the load is connected in wye, there can be no

[1] This assumes sinusoidal waves.

connection between the neutral of the load and the generator, *since no neutral connection for the alternator exists.* Although both Y- and Δ-connected loads are used, delta connection is the more common. Wye connection is the more common for alternators. Y-connected alternators are somewhat better than those connected in delta for several reasons, the most important of which are: there can be no short-circuit current in their armatures due to harmonics; for a fixed terminal voltage and output, the ratio of the amount of insulation to copper required in their armature windings is somewhat less than for delta connection; they permit grounding the neutral point of the system.

If an alternator has more than two phases, there are always two ways in which its armature windings may be connected. These are known as the *mesh* and the *star* connections and correspond to the delta and wye connections for a three-phase alternator. Alternators with more than four phases are not used. They would possess no advantage over three-phase or four-phase alternators and would have marked disadvantages from the standpoint of power transmission. They would also be more complicated. Very few four-phase alternators are built, except for special purposes such as for use in existing four-phase plants. From the standpoint of power transmission, three-phase is superior to all others. It requires only three line conductors, as against four for four-phase and six for six-phase, and for a fixed amount of power transmitted a fixed distance with a fixed line loss and fixed voltage between any two conductors (not necessarily adjacent), it requires only three-quarters as much copper as single-phase, four-phase or six-phase. From the standpoint of power transmission alone, four-phase transmission possesses no advantage over single-phase transmission. All power transmission lines are three-phase.

Relative Magnitudes and Phase Relations of Line and Phase Currents and of Line and Phase Voltages for a Balanced Three-phase System Having Sinusoidal Current and Voltage Waves.—By a balanced system is meant one in which the voltages in all phases are equal in magnitude and differ in phase by equal angles. The currents must also be equal in magnitude, and they must also differ in phase by equal angles. For a balanced system, the phase angles between the voltages and also between the currents

are equal to $\frac{360}{n}$ degrees, where n is the number of phases, except for two-phase. A balanced load is one in which the loads connected across all phases are identical.

For wye connection, there can obviously be no difference between the current in any phase and the current in the line to which the phase is connected. Line and phase currents are the same for wye connection. Line and phase voltages, however, are not the same. The voltage between any two terminals, *i.e.*, the line voltage, is the vector difference of the voltages in the phases connected between the two terminals considered. The line voltage is neither equal in magnitude to the phase voltage nor is the line voltage in phase with the phase voltage. Refer to Fig. 90. For wye connection, the same diagram may be used for either a diagram of connections or a vector diagram. Figure 90 serves for both.

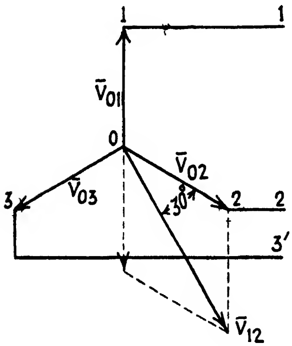


FIG. 90

Obviously,

$$\begin{aligned} \text{Line current } \bar{I}_{11}' &= \text{phase current } \bar{I}_{01} \\ \text{Line current } \bar{I}_{22}' &= \text{phase current } \bar{I}_{02} \\ \text{Line current } \bar{I}_{33}' &= \text{phase current } \bar{I}_{03} \end{aligned}$$

The line voltage $\bar{V}_{12}$ is not equal in magnitude to either $\bar{V}_{01}$ or $\bar{V}_{02}$ or in phase with either.

$$\begin{aligned} \bar{V}_{12} &= \bar{V}_{10} + \bar{V}_{02} \\ &= -\bar{V}_{01} + \bar{V}_{02} \\ V_{12} &= 2V \cos 30^\circ \\ &= \sqrt{3}V \text{ in magnitude} \end{aligned} \tag{7}$$

where V is the magnitude of the phase voltage.

$\bar{V}_{12}$ is equal in magnitude to the phase voltage multiplied by the square root of three. It differs in phase from the phase voltage $\bar{V}_{02}$ by 30 degrees and from the phase voltage $\bar{V}_{01}$ by 150 degrees.

For a balanced three-phase, Y-connected circuit, line voltage is equal in magnitude to phase voltage multiplied by the square root of three. Line voltage differs in phase from the voltages in the phases connected between the lines considered, by 30 or 150 degrees according to which of the two-phase voltages is considered.

$\bar{V}_{12}$ lags $\bar{V}_{02}$ by 30 degrees and lags $\bar{V}_{01}$ by 150 degrees (8)
$\bar{V}_{23}$ lags $\bar{V}_{03}$ by 30 degrees and lags $\bar{V}_{02}$ by 150 degrees (9)
$\bar{V}_{31}$ lags $\bar{V}_{01}$ by 30 degrees and lags $\bar{V}_{03}$ by 150 degrees (10)

If the phase rotation is opposite to that shown in Fig. 90, *i.e.*, if $\bar{V}_{02}$ leads $\bar{V}_{01}$ by 120 degrees and $\bar{V}_{03}$ leads $\bar{V}_{01}$ by 240 degrees, $\bar{V}_{12}$ leads $\bar{V}_{02}$ by 30 degrees instead of lagging it by 30 degrees, $\bar{V}_{23}$

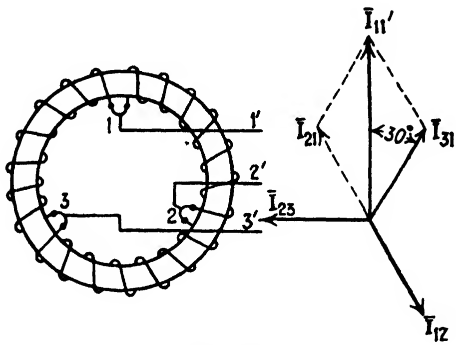


Fig. 91.

leads $\bar{V}_{03}$ by 30 degrees instead of lagging it by 30 degrees and $\bar{V}_{31}$ leads $\bar{V}_{01}$ by 30 degrees instead of lagging it by 30 degrees.

The relations existing between phase and line currents of a balanced Δ-connected system are similar to those existing between phase and line voltages of a balanced Y-connected system. Let the left- and right-hand diagrams, Fig. 91, be, respectively, a diagram of connections and a vector diagram of currents in the branches, *i.e.*, in phases, 1-2, 2-3 and 3-1 of a Δ-connected armature or a Δ-connected load.

The line current $\bar{I}_{11'}$ in line 1-1′, which is connected to the common junction point of phases 1-2 and 3-1, is equal to the vector sum of the currents in these phases both considered in a direction toward the junction point 1. This follows from the

fact that the vector sum of the currents at any junction point must be equal to zero.

$$\begin{aligned} \bar{I}_{11}' &= \bar{I}_{21} + \bar{I}_{31} \\ &= -\bar{I}_{12} + \bar{I}_{31} \end{aligned}$$

For a balanced load,

$$\begin{aligned} I_{11}' &= 2I \cos 30^\circ \\ &= \sqrt{3}I \text{ in magnitude} \end{aligned} \tag{11}$$

where I is the magnitude of the phase current.

The line current $\bar{I}_{11}'$ is equal in magnitude to the phase current multiplied by the square root of three. It differs in phase from the phase current $\bar{I}_{31}$ by 30 degrees and from the phase current $\bar{I}_{12}$ by 150 degrees.

For a balanced three-phase, Δ-connected circuit, line current is equal to phase current multiplied by the square root of three. Line current differs in phase from the currents in the phases to which the line considered is connected, by either 30 or 150 degrees according to which of the two phases is considered.

$\bar{I}_{11}'$ leads $\bar{I}_{31}$ by 30 degrees and leads $\bar{I}_{12}$ by 150 degrees (12)
$\bar{I}_{22}'$ leads $\bar{I}_{12}$ by 30 degrees and leads $\bar{I}_{23}$ by 150 degrees (13)
$\bar{I}_{33}'$ leads $\bar{I}_{23}$ by 30 degrees and leads $\bar{I}_{31}$ by 150 degrees (14)

If the phase rotation were opposite to that shown in Fig. 91, $\bar{I}_{11}'$ would lag $\bar{I}_{31}$ by 30 degrees, $\bar{I}_{22}'$ would lag $\bar{I}_{12}$ by 30 degrees and $\bar{I}_{33}'$ would lag $\bar{I}_{23}$ by 30 degrees.

In a Δ-connected system, the voltage between any pair of lines, such as 1 and 2, is obviously equal to and in phase with the voltage in the phase connected between the lines considered. It is also equal to the vector sum of the voltages in the other two phases when these are considered in the proper direction.

$$\bar{V}_{12} = \bar{V}_{13} + \bar{V}_{32} \tag{15}$$
$$\bar{V}_{23} = \bar{V}_{21} + \bar{V}_{13} \tag{16}$$
$$\bar{V}_{31} = \bar{V}_{32} + \bar{V}_{21} \tag{17}$$

For a Δ-connected system, line and phase voltages are equal.

If a single-phase load is applied between any pair of terminals of a Δ-connected alternator, the current divides between the two parallel paths formed by the armature windings inversely as their impedances. The impedance of the windings is the same

for each phase of an alternator. Since one path consists of a single phase and the other path consists of two phases in series, the impedances of the two paths between which the current divides are in the ratio of 1 to 2. Therefore, the phase across which the load is applied carries two-thirds of the current. The other two phases in series carry the remaining third. The currents in the two branches are in phase, since the ratio of the resistance to the reactance is the same in each branch.

When a balanced three-phase load is connected across the terminals of an alternator which has balanced voltages, the three line currents are equal in magnitude and differ by 120 degrees in phase. The three phase currents in the windings of the alternator are also equal in magnitude and differ by 120 degrees in phase. The phase currents are the same as the line currents for a Y-connected alternator and are equal to the line currents divided by the square root of three for a Δ-connected alternator. For the Δ-connected alternator, there is a shift in phase between line and phase currents, as is shown by equations (12), (13) and (14). The phase angle between the phase currents and phase voltages is the same for the alternator as for the load. These statements are entirely independent of whether the load and generator are connected alike, *i.e.*, both either in wye or in delta, or whether they are connected differently, *i.e.*, one in wye the other in delta.

Relative Magnitudes and Phase Relations of Line and Phase Currents and of Line and Phase Voltages of a Balanced Four-phase System Having Sinusoidal Current and Voltage Waves.—A four-phase alternator has four identical armature windings which are displaced 90 electrical degrees from one another. These windings may be connected either in star or in mesh, corresponding to the wye and delta connections for a three-phase alternator. The voltages induced in the windings of a four-phase alternator are 90 degrees apart in time phase and may, therefore, be represented by four equal vectors displaced 90 degrees from one another as shown in Fig. 92.

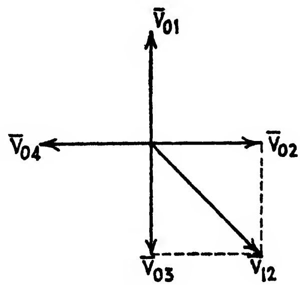


Fig. 92.

The vector expressions for the four vectors are

$$\bar{V}_{01} = V(0 + j1)$$
$$\bar{V}_{02} = V(1 + j0)$$
$$\bar{V}_{03} = V(0 - j1)$$
$$\bar{V}_{04} = V(-1 + j0)$$

where V is the magnitude of the phase voltage.

Since

$$\bar{V}_{01} + \bar{V}_{02} + \bar{V}_{03} + \bar{V}_{04} = 0 \tag{18}$$

no current flows if the four armature windings are connected in order around the armature to form a closed circuit, *i.e.*, are connected in mesh, in the same way as the armature windings of a three-phase generator are connected in delta. Instead of connecting the armature windings in mesh, they may be con-

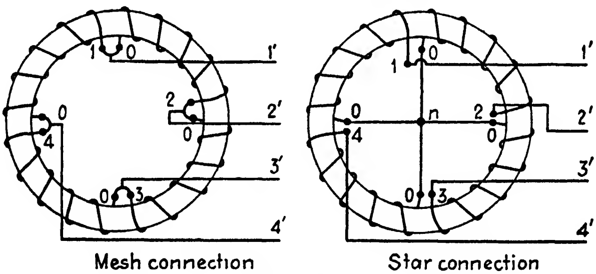


FIG. 93.

nected in star by joining their corresponding ends in the same way that the windings of a three-phase alternator are joined for wye connection.

Mesh and star four-phase connections are shown in Fig. 93. For simplicity, gramme-ring windings are shown, instead of windings like those actually used. The windings actually used are similar to those illustrated in Fig. 86, page 283, for a three-phase alternator.

A four-phase alternator must have four terminals. When connected in star, a four-phase alternator may have a fifth terminal brought out from the common junction between the windings, *i.e.*, from the neutral point.

In general, for star connection it is obvious that there can be no difference between the current in any line and the current in

the phase to which the line is connected. Line and phase currents are the same for star connection. Line and phase voltages are not the same for star connection. The voltage between any pair of line terminals, such as 1 and 2, of a star-connected alternator, *i.e.*, the line voltage, is equal to the vector sum of the voltages in the phases connected between the line terminals 1 and 2, both voltages being considered in the direction from 1 to 2 through the windings.

In general, for mesh connection, the voltage between any pair of adjacent line terminals, *i.e.*, the line voltage, is equal to the voltage in the phase to which the lines considered are connected. Line and phase voltages are equal for mesh connection. Line and phase currents, however, are not equal for mesh connection. The current per terminal, *i.e.*, the line current of a mesh-connected alternator, is equal to the vector sum of the currents in the two phases to which the line considered is connected, both currents being taken in the same direction with respect to the junction point between the phases and line, *i.e.*, both being taken either toward or away from the junction point.

Refer to Figs. 92 and 93. For a four-phase, star-connected alternator,

$$\text{Line current } \bar{I}_{11'} = \text{phase current } \bar{I}_{01} \tag{19}$$
$$\text{Line current } \bar{I}_{22'} = \text{phase current } \bar{I}_{02} \tag{20}$$
$$\text{Line current } \bar{I}_{33'} = \text{phase current } \bar{I}_{03} \tag{21}$$
$$\text{Line current } \bar{I}_{44'} = \text{phase current } \bar{I}_{04} \tag{22}$$

The line voltage $\bar{V}_{12}$ (see Fig. 92) is neither equal to nor in phase with phase voltage $\bar{V}_{01}$ or $\bar{V}_{02}$, but

$$\begin{aligned} \bar{V}_{12} &= \bar{V}_{10} + \bar{V}_{02} \\ &= -\bar{V}_{01} + \bar{V}_{02} \\ V_{12} &= 2V \cos 45° \\ &= \sqrt{2}V \text{ in magnitude} \end{aligned}$$

where V is the magnitude of the phase voltage.

The line voltage of a four-phase, star-connected alternator is equal in magnitude to the phase voltage multiplied by the square root of two. It differs in phase from the voltage in one

of the phases between the lines considered by 45 degrees and from that in the other by 135 degrees. The line and phase currents are equal.

For a mesh-connected, four-phase alternator, the current in any line, such as line 1′ (see Fig. 93), is equal to the vector sum of the currents in phases 1 and 2 both taken toward the junction point between the line and the phases.

$$\begin{aligned} \bar{I}_{11'} &= \bar{I}_{01} + \bar{I}_{20} \\ &= \bar{I}_{01} - \bar{I}_{02} \end{aligned}$$

where $\bar{I}_{01}$ and $\bar{I}_{02}$ are the currents in phases 01 and 02, respectively.

For a balanced load,

$$\begin{aligned} I_{11'} &= 2I \cos 45° \\ &= \sqrt{2}I \text{ in magnitude} \end{aligned} \qquad (23)$$

where I is the magnitude of the phase currents.

The line current of a four-phase, mesh-connected alternator, carrying a balanced load, is equal in magnitude to the phase current multiplied by the square root of two. There is a phase difference between the current in any line and in the phases to which it connects of 45 or 135 degrees, according to which of the two phases is considered. The line and phase voltages of a mesh-connected, four-phase alternator are equal.

Relative Magnitudes of Line and Phase Currents and Line and Phase Voltages for Balanced Star- and Mesh-connected, N-phase Systems Having Sinusoidal Current and Voltage Waves. The line voltages of any balanced mesh-connected system are always equal to the phase voltages. The line and phase currents of any balanced star-connected system are always equal. These relations, which have already been stated, are obvious from an inspection of Figs. 89 and 90. For any balanced n-phase system, the phase voltages and also the phase currents differ in time phase by $\frac{360}{n}$ degrees, where n is the number of phases, except for two-phase.

The complex expressions for the phase voltages of an n-phase system are

$$\bar{V}_{01} = V\{\cos 0° - j \sin 0°\} \tag{24}$$

$$\bar{V}_{02} = V\left\{\cos \frac{360°}{n} - j \sin \frac{360°}{n}\right\} \tag{25}$$

$$\bar{V}_{03} = V\left\{\cos 2\,\frac{360°}{n} - j \sin 2\,\frac{360°}{n}\right\} \tag{26}$$

. . . .

. . . .

$$\bar{V}_{0n} = V\left\{\cos (n-1)\,\frac{360°}{n} - j \sin (n-1)\,\frac{360°}{n}\right\}$$

where V is the magnitude of the phase voltages. If the system is star-connected, the voltage $\bar{V}_{12}$ between line terminals 1 and 2 is

$$\begin{aligned}\bar{V}_{12} &= \bar{V}_{10} + \bar{V}_{02} \\ &= -\bar{V}_{01} + \bar{V}_{02}\end{aligned}$$

The voltages $\bar{V}_{01}$ and $\bar{V}_{02}$ are $\frac{360}{n}$ degrees apart in time phase. Therefore $-\bar{V}_{01}$ and $\bar{V}_{02}$ are $\left(180 - \frac{360}{n}\right)$ degrees apart in phase. Hence for a balanced system, $\bar{V}_{12}$ is equal to the vector sum of two equal voltages which differ in phase by $\left(180 - \frac{360}{n}\right)$ degrees.

$$V_{12} = 2V \cos \frac{1}{2}\left(180° - \frac{360°}{n}\right) \text{ in magnitude}$$

$$V_{line} = 2V_{star} \cos \frac{1}{2}\left(180° - \frac{360°}{n}\right) \text{ in magnitude} \tag{27}$$

A similar relation can be shown to exist between the magnitudes of the line and phase currents of a balanced mesh-connected system. For a mesh-connected system,

$$I_{line} = 2I_{mesh} \cos \frac{1}{2}\left(180° - \frac{360°}{n}\right) \text{ in magnitude} \tag{28}$$

The following relations between the magnitudes of the line and phase currents and between the line and phase voltages for balanced systems may be obtained from equations (27) and (28) and the general statements that precede.

Number of phases	Star connection		Mesh connection	
	Line current equals	Line voltage equals	Line current equals	Line voltage equals
3	I_{phase}	$\sqrt{3}\ V_{phase}$	$\sqrt{3}\ I_{phase}$	V_{phase}
4	I_{phase}	$\sqrt{2}\ V_{phase}$	$\sqrt{2}\ I_{phase}$	V_{phase}
6	I_{phase}	V_{phase}	I_{phase}	V_{phase}
12	I_{phase}	$0.518\ V_{phase}$	$0.518\ I_{phase}$	V_{phase}

Example.—A certain experimental, 60-cycle, alternating-current generator has six identical armature windings which are displaced 60 electrical degrees from one another. When the alternator is driven at rated speed and has normal excitation, the voltage generated in each winding is 50 volts. What are the line and phase voltages of this alternator when the armature windings are connected for six-phase, for three-phase and for single-phase?

The complex expressions for the voltages in the armature windings may be written

$$\begin{aligned}
\bar{V}_{01} &= 50\ (\cos 0° - j \sin 0°) &&= 50.0 - j0 \\
\bar{V}_{02} &= 50\ (\cos 60° - j \sin 60°) &&= 25.0 - j43.3 \\
\bar{V}_{03} &= 50\ (\cos 120° - j \sin 120°) &&= -25.0 - j43.3 \\
\bar{V}_{04} &= 50\ (\cos 180° - j \sin 180°) &&= -50.0 - j0 \\
\bar{V}_{05} &= 50\ (\cos 240° - j \sin 240°) &&= -25.0 + j43.3 \\
\bar{V}_{06} &= 50\ (\cos 300° - j \sin 300°) &&= 25.0 + j43.3
\end{aligned}$$

Six-phase

One winding is used for each phase.

Star Connection

$$\begin{aligned}
\bar{V}_{12} &= \bar{V}_{10} + \bar{V}_{02} \\
&= -\bar{V}_{01} + \bar{V}_{02} \\
&= -(50 - j0) + (25.0 - j43.3) \\
&= -25.0 - j43.3 \\
V_{12} &= \sqrt{(-25.0)^2 + (-43.3)^2} = 50 \text{ volts} \\
V_{line} &= 50 \text{ volts}
\end{aligned}$$

Mesh Connection

(Line and phase voltages are equal)

$V_{line} = 50$ volts

Three-phase

Two windings are used in series for each phase. The phase voltages are

$$\bar{V}_{oa} = \bar{V}_{01} + \bar{V}_{02} = (50.0 - j0) + (25.0 - j43.3)$$
$$= 75.0 - j43.3$$
$$\bar{V}_{ob} = \bar{V}_{03} + \bar{V}_{04} = (-25.0 - j43.3) + (-50.0 - j0)$$
$$= -75.0 - j43.3$$
$$\bar{V}_{oc} = \bar{V}_{05} + \bar{V}_{06} = (-25.0 + j43.3) + (25.0 + j43.3)$$
$$= 0 + j86.6$$

Star Connection

$$\bar{V}_{ab} = \bar{V}_{ao} + \bar{V}_{ob}$$
$$= -\bar{V}_{oa} + \bar{V}_{ob}$$
$$= -(75.0 - j43.3) + (-75.0 - j43.3)$$
$$= -150.0 - j0$$
$$V_{ab} = \sqrt{(-150)^2 + (0)^2} = 150 \text{ volts}$$
$$V_{line} = 150 \text{ volts}$$

Mesh Connection—
(Line and phase voltages are equal)

$$V_{line} = \sqrt{(75.0)^2 + (-43.3)^2}$$
$$= 86.6 \text{ volts}$$

Single-phase

If all six windings are connected in series, their resultant voltage is zero. For single-phase, the windings may be divided into two groups of three each. The windings in each group are connected in series and then the two groups paralleled.

First group, $\bar{V}' = \bar{V}_{01} + \bar{V}_{02} + \bar{V}_{03}$

$$= (50.0 - j0) + (25.0 - j43.3) + (-25.0 - j43.3)$$
$$= 50.0 - j86.6$$

Second group, $\bar{V}'' = \bar{V}_{04} + \bar{V}_{05} + \bar{V}_{06}$

$$= (-50.0 - j0) + (-25.0 + j43.3) + (25.0 + j43.3)$$
$$= -50.0 + j86.6$$

The voltages $\bar{V}'$ and $\bar{V}''$ are equal in magnitude but opposite in phase. By reversing the connections of one group of windings, the voltages in the two groups of windings are brought into phase. The two groups of windings are then paralleled, giving a single-phase alternator with a voltage of

$$V_{line} = \sqrt{(-50.0)^2 + (86.6)^2}$$
$$= 100 \text{ volts}$$

and a current capacity of twice the current capacity of each winding.

Example of a Balanced Δ-connected Load.—Three equal impedances, each having a resistance of 10 ohms and an inductive reactance of 15 ohms at 60 cycles, are connected in delta across a balanced three-phase, 230-volt circuit. What is the line current and what is the total power absorbed by the load?

The magnitude of the phase currents is

$$I_{phase} = \frac{V_{phase}}{\sqrt{r^2 + x^2}}$$
$$= \frac{230}{\sqrt{(10)^2 + (15)^2}} = \frac{230}{18.03}$$
$$= 12.75 \text{ amperes}$$

$$I_{line} = \sqrt{3} \times 12.75$$
$$= 22.08 \text{ amperes}$$

$$P_{phase} = (I_{phase})^2 \times r_{phase}$$
$$= (12.75)^2 \times 10$$
$$= 1626 \text{ watts}$$

$$P_{total} = 3 \times P_{phase}$$
$$= 3 \times 1626$$
$$= 4878 \text{ watts}$$

Example of an Unbalanced Δ-connected Load.—Three impedances, z_{12}, z_{23} and z_{31}, are connected in delta across the lines 1–2, 2–3 and 3–1, respectively, of a 220-volt, three-phase, 60-cycle circuit having balanced voltages. Each of the impedances has a resistance of 5 ohms. Impedances z_{12} and z_{23} have inductive reactances at 60 cycles of 5 and 10 ohms, respectively. Impedance z_{31} has a capacitive reactance at 60 cycles of 10 ohms. If the phase order of the voltages $\bar{V}_{12}$, $\bar{V}_{23}$ and $\bar{V}_{31}$ between the terminals 1–2, 2–3 and 3–1 of the circuit is clockwise, *i.e.*, if $\bar{V}_{12}$ leads $\bar{V}_{23}$ and $\bar{V}_{23}$ leads $\bar{V}_{31}$, what are the three line currents? What is the total power consumed by the Δ-connected load?

A diagram of connections is shown in Fig. 94.

Take the voltage $\bar{V}_{12}$ between lines 1 and 2 as the axis of reference. The complex expressions for the three line voltages are then

$$\bar{V}_{12} = 220\ (1 + j0) = 220 + j0$$

$$\bar{V}_{23} = 220\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -110 - j190.5$$

$$\bar{V}_{31} = 220\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = -110 + j190.5$$

The phase currents, *i.e.*, the currents in the branches of the Δ-connected load, may be found in complex by dividing the

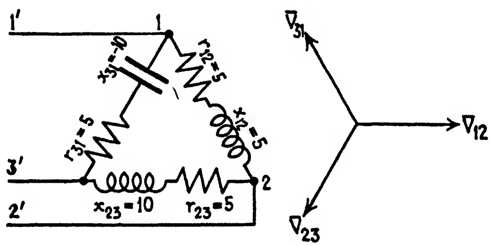


FIG. 94.

phase voltages in complex by the impedances also expressed in complex. The phase currents are

$$\bar{I}_{12} = \frac{220 + j0}{5 + 5j}$$
$$= 22 - j22$$

$$\bar{I}_{23} = \frac{-110 - j190.5}{5 + j10}$$
$$= -19.64 + j1.18$$

$$\bar{I}_{31} = \frac{-110 + j190.5}{5 - j10}$$
$$= -19.64 - j1.18$$

From Fig. 94, it is obvious that the line currents are

$$\bar{I}_{1'1} = \bar{I}_{12} + \bar{I}_{13} = \bar{I}_{12} - \bar{I}_{31}$$
$$= (22 - j22) - (-19.64 - j1.18)$$
$$= 41.64 - j20.82$$

$$I_{1'1} = \sqrt{(41.64)^2 + (-20.82)^2}$$
$$= 46.55 \text{ amperes}$$

$$\begin{aligned}
\bar{I}_{2'2} &= \bar{I}_{23} + \bar{I}_{21} = \bar{I}_{23} - \bar{I}_{12} \\
&= (-19.64 + j1.18) - (22 - j22) \\
&= -41.64 + j23.18 \\
I_{2'2} &= \sqrt{(-41.64)^2 + (23.18)^2} \\
&= 47.65 \text{ amperes} \\
\bar{I}_{3'3} &= \bar{I}_{31} + \bar{I}_{32} = \bar{I}_{31} - \bar{I}_{23} \\
&= (-19.64 - j1.18) - (-19.64 + j1.18) \\
&= 0 - j2.36 \\
I_{3'3} &= \sqrt{(0)^2 + (-2.36)^2} \\
&= 2.36 \text{ amperes}
\end{aligned}$$

The total power is equal to the sum of the losses due to the resistances in the three phases:

$$\begin{aligned}
P_{total} &= (I_{12})^2 \times r_{12} + (I_{23})^2 \times r_{23} + (I_{31})^2 \times r_{31} \\
&= \{(22)^2 + (22)^2\} \times 5 + \{(19.64)^2 + (1.18)^2\} \times 5 \\
&\qquad + \{(19.64)^2 + (1.18)^2)\} \times 5 \\
&= 4840 + 1936 + 1936 \\
&= 8712 \text{ watts}
\end{aligned}$$

If the opposite phase order, *i.e.*, counter-clockwise, had been assumed, the total power would have been the same as for clockwise-phase order, but the three line currents would have been different.

Balanced and Unbalanced Y-connected Loads.—Balanced and unbalanced Y-connected loads can best be taken up after the application of Kirchhoff's laws to alternating-current circuits has been considered. They will be considered in the next chapter.

CHAPTER X

KIRCHHOFF'S LAWS APPLIED TO POLYPHASE CIRCUITS AND EQUIVALENT Y-CONNECTED AND Δ-CONNECTED CIRCUITS

Example of the Application of Kirchhoff's Laws to a Simple Three-phase Circuit.—For a statement of Kirchhoff's laws, see Chapter VII, page 212. Before attempting to solve any problem involving Kirchhoff's laws, a diagram of connections should be made. Each corner or junction point of this diagram should be numbered or lettered. Double-subscript notation should be used. (See page 284.) Three impedances, $\bar{z}_{oa} = 10 + j0$, $\bar{z}_{ob} = 1 + j10$ and $\bar{z}_{oc} = 0 - j10$, are connected in wye to the lines a, b and c, respectively, of a three-phase, 230-volt circuit with balanced voltages (Fig. 95). If the voltage drop $\bar{V}_{ab}$ between lines a and b leads the voltage drop $\bar{V}_{bc}$, what are the three line currents and the voltage drops between the lines and common junction point of the impedances? If $\bar{V}_{ab}$ is taken lagging $\bar{V}_{bc}$ instead of leading it, a different solution results.

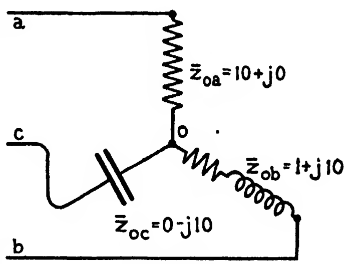


FIG. 95.

If the voltage $\bar{V}_{ab}$ is taken as the axis of reference, the vector expressions for the three line voltages are

$$\bar{V}_{ab} = 230 + j0 \tag{1}$$
$$\bar{V}_{bc} = -115 - j199.2 \tag{2}$$
$$\bar{V}_{ca} = -115 + j199.2 \tag{3}$$

Since two subscripts are not necessary on Y-connected impedances in order to indicate to which phases they belong, only one subscript will be used with each impedance in what follows. The impedances z_{ao}, z_{bo} and z_{co} on the figure are written z_a, z_b and

$\bar{z}_c$, respectively. Double subscripts are needed on mesh-connected impedances to make them definite.

From Kirchhoff's laws,

$$\bar{V}_{ab} = \bar{I}_{ao}\bar{z}_a + \bar{I}_{ob}\bar{z}_b \quad (4)$$
$$\bar{V}_{bc} = \bar{I}_{bo}\bar{z}_b + \bar{I}_{oc}\bar{z}_c \quad (5)$$
$$\bar{V}_{ca} = \bar{I}_{co}\bar{z}_c + \bar{I}_{oa}\bar{z}_a \quad (6)$$
$$\bar{I}_{ao} + \bar{I}_{bo} + \bar{I}_{co} = 0 \quad (7)$$

Equations (4), (5) and (6) are not three independent simultaneous equations, since any one of the three is equal to the negative of the vector sum of the other two. This follows from the fact that the vector sum of the line voltages of any polyphase circuit, taken in order, must be zero. $\bar{V}_{ab} + \bar{V}_{bc} + \bar{V}_{ca} = 0$.

Equations (4), (6) and (7) will be used in solving for $\bar{I}_{bo}$.

Add equations (4) and (6), remembering that reversing the subscripts on a vector is the same as reversing its sign.

$$\bar{V}_{ab} + \bar{V}_{ca} = \bar{I}_{ob}\bar{z}_b + \bar{I}_{co}\bar{z}_c \quad (8)$$

Substitute $\bar{I}_{oa}$ from equation (7) in equation (6), then

$$\begin{aligned} \bar{V}_{ca} &= \bar{I}_{co}\bar{z}_c + (\bar{I}_{bo} + \bar{I}_{co})\bar{z}_a \\ &= \bar{I}_{co}(\bar{z}_c + \bar{z}_a) + \bar{I}_{bo}\bar{z}_a \end{aligned} \quad (9)$$

Substitute $\bar{I}_{co}$ from equation (8) in equation (9), then

$$\bar{V}_{ca} = (\bar{z}_c + \bar{z}_a)\frac{(\bar{V}_{ab} + \bar{V}_{ca}) - \bar{I}_{ob}\bar{z}_b}{\bar{z}_c} + \bar{I}_{bo}\bar{z}_a$$

$$\begin{aligned} \bar{V}_{ca}\bar{z}_c = &-\bar{I}_{ob}\bar{z}_b\bar{z}_c - \bar{I}_{ob}\bar{z}_b\bar{z}_a + \bar{I}_{bo}\bar{z}_a\bar{z}_c \\ &+ \bar{V}_{ab}\bar{z}_c + \bar{V}_{ca}\bar{z}_c + \bar{V}_{ab}\bar{z}_a + \bar{V}_{ca}\bar{z}_a \end{aligned}$$

$$\bar{I}_{ob} = \frac{(\bar{V}_{ab} + \bar{V}_{ca})\bar{z}_a + \bar{V}_{ab}\bar{z}_c}{\bar{z}_a\bar{z}_b + \bar{z}_b\bar{z}_c + \bar{z}_c\bar{z}_a} \quad (10)$$

Since $\bar{V}_{ab} + \bar{V}_{bc} + \bar{V}_{ca} = 0$,

$$\bar{I}_{ob} = \frac{\bar{V}_{ab}\bar{z}_c - \bar{V}_{bc}\bar{z}_a}{\bar{z}_a\bar{z}_b + \bar{z}_b\bar{z}_c + \bar{z}_c\bar{z}_a} \quad (11)$$

Since the circuit is symmetrical, the equations for the other currents can be found by advancing each of the terms in equation (10) one phase for I_{oc} and two phases for I_{oa}.

$$\bar{I}_{oc} = \frac{\bar{V}_{bc}\bar{z}_a - \bar{V}_{ca}\bar{z}_b}{\bar{z}_b\bar{z}_c + \bar{z}_c\bar{z}_a + \bar{z}_a\bar{z}_b} \quad (12)$$

$$\bar{I}_{oa} = \frac{\bar{V}_{ca}\bar{z}_b - \bar{V}_{ab}\bar{z}_c}{\bar{z}_c\bar{z}_a + \bar{z}_a\bar{z}_b + \bar{z}_b\bar{z}_c} \quad (13)$$

The vector values of the three currents can be found by substituting the vector values for the letters representing voltages and impedances in equations (11), (12) and (13).

$$\bar{I}_{oa} = \frac{(-115+j199.2)(1+j10)-(230+j0)(0-j10)}{(0-j10)(10+j0)+(10+j0)(1+j10)+(1+j10)(0-j10)}$$

$$= \frac{-2107+j1349.2}{110-j10} = -20.10+j10.44 \text{ vector amperes}$$

$$\bar{I}_{ob} = \frac{(230+j0)(0-j10)-(-115-j199.2)(10+j0)}{110-j10}$$

$$= \frac{1150-j308}{110-j10} = 10.62-j1.83 \text{ vector amperes}$$

$$\bar{I}_{oc} = -\bar{I}_{oa} - \bar{I}_{ob} = (20.10-j10.44)+(-10.62+j1.83)$$

$$= 9.48-j8.61 \text{ vector amperes}$$

$$\bar{I}_{oa} = -20.10+j10.44 \text{ vector amperes}$$
$$\bar{I}_{ob} = 10.62-j1.83 \text{ vector amperes}$$
$$\bar{I}_{oc} = 9.48-j8.61 \text{ vector amperes}$$

$$\bar{V}_{oa} = \bar{I}_{oa}\bar{z}_{oa}$$
$$= (-20.10+j10.44)(10+j0)$$
$$= -201.0+j104.4$$

$$\bar{V}_{ob} = \bar{I}_{ob}\bar{z}_{ob}$$
$$= (10.62-j1.83)(1+j10)$$
$$= 28.92+j104.4$$

$$\bar{V}_{oc} = \bar{I}_{oc}\bar{z}_{oc}$$
$$= (9.48-j8.61)(0-j10)$$
$$= -86.1-j94.8$$

As a check, the voltages $\bar{V}_{oa}$, $\bar{V}_{ob}$ and $\bar{V}_{oc}$ can be combined to give the voltages between lines.

$$\bar{V}_{ab} = \bar{V}_{ao} + \bar{V}_{ob} = 201.0-j104.4+28.92+j104.4$$
$$= 229.9+j0$$

$$\bar{V}_{bc} = \bar{V}_{bo} + \bar{V}_{oc} = -28.92-j104.4-86.1-j94.8$$
$$= -115.0-j199.2$$

$$\bar{V}_{ca} = \bar{V}_{co} + \bar{V}_{oa} = 86.0+j94.8-201.0+j104.4$$
$$= -115.0+j199.2$$

Balanced Y-connected Loads Connected across Three-phase Circuits Having Balanced Voltages.—When three equal impedances are connected in wye across a three-phase system whose

voltages are balanced, the common junction of the impedances is the true neutral point of the three-phase system. The currents may, therefore, be found both in phase and in magnitude by dividing the wye voltages of the system by the impedances, both voltages and impedances being expressed in complex.

Refer to Fig. 96.

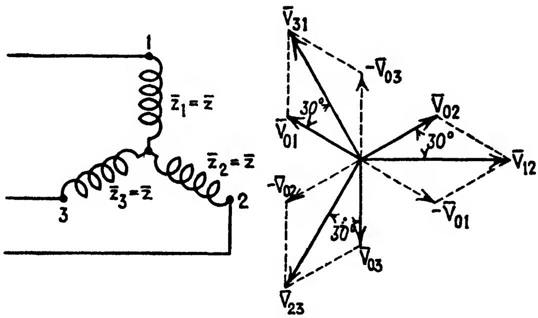


FIG. 96.

Let $\bar{z}_1 = \bar{z}$, $\bar{z}_2 = \bar{z}$ and $\bar{z}_3 = \bar{z}$ be three equal impedances connected in wye across the terminals of a three-phase system having balanced voltages. Let the line voltages be $\bar{V}_{12}$, $\bar{V}_{23}$ and $\bar{V}_{31}$. The diagram of connections is shown in the left-hand half of Fig. 96. The vector diagram of voltages is shown in the right-hand half. $\bar{V}_{01}$, $\bar{V}_{02}$ and $\bar{V}_{03}$ are the phase voltages of a balanced Y-connected system having line voltages equal to $\bar{V}_{12}$, $\bar{V}_{23}$ and $\bar{V}_{31}$. The voltage $\bar{V}_{12}$, between lines 1 and 2, is taken as the axis of reference.

The vector expressions for the three line voltages are

$$\bar{V}_{12} = V\,(1 + j0) \tag{14}$$

$$\bar{V}_{23} = V\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \tag{15}$$

$$\bar{V}_{31} = V\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \tag{16}$$

where V is the magnitude of the voltages between the lines 1–2, 2–3 and 3–1.

Applying Kirchhoff's laws to the circuit gives

$$\bar{V}_{12} = \bar{I}_{10}\bar{z} + \bar{I}_{02}\,\bar{z} \tag{17}$$

$$\bar{V}_{23} = \bar{I}_{20}\bar{z} + \bar{I}_{03}\,\bar{z} \tag{18}$$

$$\bar{V}_{31} = \bar{I}_{30}\bar{z} + \bar{I}_{01}\,\bar{z} \tag{19}$$

$$\bar{I}_{01} + \bar{I}_{02} + \bar{I}_{03} = 0 \tag{20}$$

From equations (17) and (18),

$$\bar{I}_{10} = \frac{\bar{V}_{12} - \bar{I}_{02}\,\bar{z}}{\bar{z}} = \frac{\bar{V}_{12}}{\bar{z}} - \bar{I}_{02} \tag{21}$$

$$\bar{I}_{03} = \frac{\bar{V}_{23} - \bar{I}_{20}\,\bar{z}}{\bar{z}} = \frac{\bar{V}_{23}}{\bar{z}} - \bar{I}_{20} \tag{22}$$

Substituting the values of $\bar{I}_{10}$ and $\bar{I}_{03}$ from equations (21) and (22) in equation (20) gives

$$-\frac{\bar{V}_{12}}{\bar{z}} + \bar{I}_{02} + \bar{I}_{02} + \frac{\bar{V}_{23}}{\bar{z}} - \bar{I}_{20} = 0$$

$$3\,\bar{I}_{02} = \frac{\bar{V}_{12} - \bar{V}_{23}}{\bar{z}} \tag{23}$$

Replacing $\bar{V}_{12}$ and $\bar{V}_{23}$ by their complex expressions from equations (14) and (15) gives

$$\bar{I}_{02} = \frac{V}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} + j\,\frac{1}{2}\right) \times \frac{1}{\bar{z}}$$

$$= \frac{V}{\sqrt{3}}\,(\cos 30° + j \sin 30°) \times \frac{1}{\bar{z}} \tag{24}$$

$$I_{line} = \frac{\dfrac{V}{\sqrt{3}}\sqrt{(\cos^2 30° + \sin^2 30°)}}{z}$$

$$= \frac{V}{\sqrt{3}\,z} \text{ in magnitude} \tag{25}$$

where V is the magnitude of the voltages between lines, *i.e.*, of the line voltages.

The delta voltage or line voltage of a balanced three-phase system is equal in magnitude to the wye voltage multiplied by the square root of three. [See equation (7), page 321.] It is displaced from the wye voltage by 30 degrees. Conversely, the wye voltage is equal in magnitude to the delta voltage divided by the square root of three and is displaced from the delta voltage by 30 degrees.

By referring to the vector diagram in Fig. 96, it is evident that $\dfrac{V}{\sqrt{3}}$ (cos 30° + j sin 30°) is the wye voltage of the system

for phase 2. The current in phase 2 is, therefore, equal in magnitude to and in phase with the current obtained by dividing the wye voltage for phase 2, expressed in complex, by the impedance of the phase, also expressed in complex.

Expressions similar to equations (24) and (25) may be found for the other line currents.

It follows from equations (24) and (25) that the phase or line currents for a balanced Y-connected load, which is connected to a three-phase system whose voltages are balanced, are equal in magnitude to the line voltage divided by the square root of three and by the phase impedance. It is equal to the wye voltage of the system divided by the phase impedance. The vector expression for the current in any phase, such as phase 1, is equal to the wye voltage for phase 1, expressed in complex, divided by the phase impedance, also expressed in complex. It must not be forgotten that the preceding statements are true *only* for a balanced system. If either the voltages or the impedances are unbalanced, the statements are not correct.

Example of a Balanced Y-connected Load.—Three equal impedances, each having a resistance of 10 ohms and an inductive reactance of 15 ohms, are connected in wye to a 230-volt, three-phase system whose voltages are balanced. What are the line currents and the total power taken by the load?

The wye voltage of the system is

$$V_Y = \frac{V_{line}}{\sqrt{3}}$$

$$I_{line} = I_Y = \frac{V_Y}{z}$$

$$= \frac{\frac{230}{\sqrt{3}}}{\sqrt{(10)^2 + (15)^2}} = \frac{132.8}{18.03}$$

$$= 7.36 \text{ amperes}$$

$$\text{Total power} = P_0 = 3 \times (I_{phase})^2 \times r_{phase}$$

$$= 3 \times (7.36)^2 \times 10$$

$$= 1625 \text{ watts}$$

An Example Involving Balanced Y- and Δ-connected Loads in Parallel across a Three-phase System Whose Voltages Are Balanced.—Three equal impedances, each having 5 ohms

resistance and 5 ohms inductive reactance, are connected in wye across a 230-volt, three-phase circuit whose voltages are balanced. Three other equal impedances, each having 10 ohms resistance and 5 ohms capacitive reactance, are connected in delta across the same circuit. What is the resultant line current? What is the total power taken by the two loads in parallel?

The complex expressions for the impedances are

$$\bar{z}_Y = 5 + j5$$
$$\bar{z}_\Delta = 10 - j5$$

The diagram of connections and a vector diagram of the wye and delta voltages are shown in Fig. 97. The voltage $\bar{V}_{01}$ between neutral and line 1 is taken along the j axis.

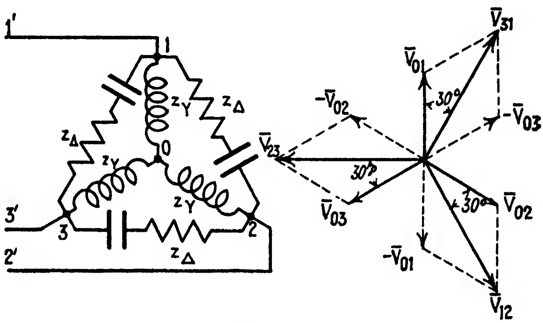


FIG. 97.

Since the load is balanced, it is necessary to consider the current in one line only. According to Kirchhoff's laws, line current $\bar{I}_{11'} = \bar{I}_{01} + \bar{I}_{21} + \bar{I}_{31}$.

The vector expressions for the voltages $\bar{V}_{01}$, $\bar{V}_{12}$ and $\bar{V}_{31}$ are

$$\begin{aligned}
\bar{V}_{01} &= \frac{230}{\sqrt{3}}(0 + j1) \\
&= 0 + j132.8 \\
\bar{V}_{12} &= \bar{V}_{10} + \bar{V}_{02} = -\bar{V}_{01} + \bar{V}_{02} \\
&= -\frac{230}{\sqrt{3}}(0 + j1) + \frac{230}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) \\
&= 230\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \\
&= 115 - j199.2
\end{aligned}$$

$$
\begin{aligned}
\bar{V}_{31} &= \bar{V}_{30} + \bar{V}_{01} = -\bar{V}_{03} + \bar{V}_{01} \\
&= -\frac{230}{\sqrt{3}}\left(-\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + \frac{230}{\sqrt{3}}(0 + j1) \\
&= 230\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \\
&= 115 + j199.2
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_{21} &= \frac{\bar{V}_{21}}{\bar{z}_\Delta} = \frac{-\bar{V}_{12}}{\bar{z}_\Delta} \\
&= \frac{-115 + j199.2}{10 - j5} \\
&= \frac{-115 + j199.2}{10 - j5} \times \frac{10 + j5}{10 + j5} \\
&= -17.17 + j11.34
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_{31} &= \frac{\bar{V}_{31}}{\bar{z}_\Delta} \\
&= \frac{115 + j199.2}{10 - j5} \\
&= \frac{115 + j199.2}{10 - j5} \times \frac{10 + j5}{10 + j5} \\
&= 1.232 + j20.54
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_{01} &= \frac{\bar{V}_{01}}{\bar{z}_Y} \\
&= \frac{0 + j132.8}{5 + j5} \\
&= \frac{0 + j132.8}{5 + j5} \times \frac{5 - j5}{5 - j5} \\
&= 13.28 + j13.28
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_{11'} &= \bar{I}_{01} + \bar{I}_{21} + \bar{I}_{31} \\
&= (13.28 + j13.28) + (-17.17 + j11.34) \\
&\qquad + (1.232 + j20.54) \\
&= -2.66 + j45.16
\end{aligned}
$$

$$
\begin{aligned}
I_{11'} &= \sqrt{(-2.66)^2 + (45.16)^2} \\
&= 45.24 \text{ amperes}
\end{aligned}
$$

$$
\begin{aligned}
\text{Total power} &= P_0 = 3I_\Delta^2 r_\Delta + 3I_Y^2 r_Y \\
&= 3\{(-17.17)^2 + (11.34)^2\} \times 10 \\
&\qquad + 3\{(13.28)^2 + (13.28)^2\} \times 5 \\
&= 17{,}990 \text{ watts}
\end{aligned}
$$

This problem could have been solved somewhat more easily by replacing the Δ-connected load by its equivalent Y-connected load. Equivalent Y- and Δ-connected loads will now be considered.

Equivalent Unbalanced Y- and Δ-connected, Three-phase Circuits.—If an equivalent three-phase system is defined as one which takes the same line currents at the same line voltages with the same phase relations between the line currents and line voltages as the three-phase system it replaces, then any unbalanced delta system of *currents* can be replaced by just *one* equivalent wye system of currents; but any unbalanced delta system of *voltages* can be replaced by an *infinite number* of wye systems of voltages. Fixing *either* the voltage or the constants of one of the equivalent wye branches fixes *both* the voltages and constants of the other branches. Conversely, any unbalanced wye system of *voltages* can be replaced by just *one* equivalent delta system of voltages, but any unbalanced wye system of *currents* can be replaced by an *infinite number* of delta systems of currents. Fixing *either* the current or the constants of one of the equivalent delta branches fixes both the currents and constants of the other branches.

If the above definition of an equivalent three-phase system is further qualified by the condition that the equivalent system must not only be equivalent as a whole, but also equivalent between each pair of mains, *i.e.*, if any main is opened the system must take the same current from the remaining two mains with the same phase relation between this current and the line voltage as the three-phase system it replaces would have taken under like conditions, then there is just one delta system which can exactly replace a wye system, and, conversely, there is just one wye system which can exactly replace a delta system. That any Δ-connected system can be replaced, so far as voltages alone are concerned, by an infinite number of Y-connected systems is evident by referring to Fig. 98. Let $\bar{V}_{ab}$, $\bar{V}_{bc}$ and $\bar{V}_{ca}$ be the voltages of the Δ-connected system and $\bar{V}_{oa}$, $\bar{V}_{ob}$ and

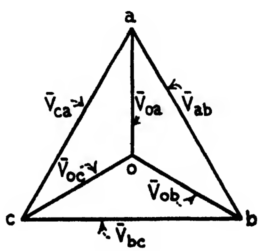


FIG. 98

$\bar{V}_{oc}$ the voltages of a Y-connected system which has the same line voltages. Obviously, any three wye vectors having their ends at a, b and c can replace the three delta voltages. The common point o, from which the three wye vectors are drawn, may be anywhere either within or without the delta.

Relations between the Constants of Unbalanced Δ- and Equivalent Y-connected, Three-phase Circuits.—Refer to Fig. 99. Let line c be open. For the Δ-connected system, z_{ab} is in parallel with z_{bc} and z_{ca} in series. For the Y-connected system, $\bar{z}_{oa}$ and

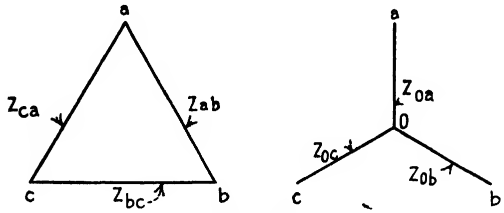


FIG. 99

z_{ob} are in series. For equivalence between lines a and b the following relation must hold:

$$\bar{z}'_{ab} = \frac{\bar{z}_{ab}(\bar{z}_{bc} + \bar{z}_{ca})}{\bar{z}_{ab} + \bar{z}_{bc} + \bar{z}_{ca}} = \bar{z}_{oa} + \bar{z}_{ob} \tag{26}$$

With line a open,

$$\bar{z}'_{bc} = \frac{\bar{z}_{bc}(\bar{z}_{ca} + \bar{z}_{ab})}{\bar{z}_{ab} + \bar{z}_{bc} + \bar{z}_{ca}} = \bar{z}_{ob} + \bar{z}_{oc} \tag{27}$$

With line b open,

$$\bar{z}'_{ca} = \frac{\bar{z}_{ca}(\bar{z}_{ab} + \bar{z}_{bc})}{\bar{z}_{ab} + \bar{z}_{bc} + \bar{z}_{ca}} = \bar{z}_{oc} + \bar{z}_{oa} \tag{28}$$

Subtracting equation (28) from equation (26) gives

$$\bar{z}_{ob} - \bar{z}_{oc} = \frac{\bar{z}_{ab}\bar{z}_{bc} - \bar{z}_{ca}\bar{z}_{bc}}{\bar{z}_{ab} + \bar{z}_{bc} + \bar{z}_{ca}} \tag{29}$$

Adding equations (27) and (29) gives

$$\bar{z}_{ob} = \frac{\bar{z}_{ab}\bar{z}_{bc}}{\bar{z}_{ab} + \bar{z}_{bc} + \bar{z}_{ca}} \tag{30}$$

The impedances $\bar{z}_{oc}$ and $\bar{z}_{oa}$ can be found in a similar manner.

Since the circuits are symmetrical, equations (31) and (32) may be found directly from equation (30). Equation (31) is

found from equation *(30)* by advancing all of the terms in equation (30) one phase. Similarly, equation (32) is found by advancing all of the terms in equation (30) two phases.

$$\bar{z}_{oc} = \frac{\bar{z}_{bc}\bar{z}_{ca}}{\bar{z}_{ab} + \bar{z}_{bc} + \bar{z}_{ca}} \tag{31}$$

$$\bar{z}_{oa} = \frac{\bar{z}_{ca}\bar{z}_{ab}}{\bar{z}_{ab} + \bar{z}_{bc} + \bar{z}_{ca}} \tag{32}$$

All impedances must be expressed in their complex form.

The neutral of an unbalanced three-phase, Δ-connected circuit may be considered to be the neutral point of the equivalent Y-connected circuit as determined by the impedances $\bar{z}_{oa}$, $\bar{z}_{ob}$ and $\bar{z}_{oc}$, given by the preceding equations.

Although it is occasionally convenient in certain problems to be able to replace an unbalanced Δ-connected circuit by its equivalent unbalanced Y-connected circuit, little is saved, as a rule, in the amount of time and labor involved in obtaining a complete solution. The work of changing the given constants to the constants of the equivalent circuit is usually as great as the labor saved in solving the new equivalent unbalanced circuit over what would have been required to solve the original circuit. The conditions are very different when the loads are balanced, as the transfer from a balanced delta connection to the equivalent balanced wye connection, or *vice versa,* can be made quickly and easily merely by the use of the factor 3.

The equivalent Δ-connected impedances in terms of the Y-connected impedances can be found from equations (26), (27) and (28). These impedances are

$$\bar{z}_{ab} = \frac{\bar{z}_{oa}\bar{z}_{ob} + \bar{z}_{ob}\bar{z}_{oc} + \bar{z}_{oc}\bar{z}_{oa}}{\bar{z}_{oc}} \tag{33}$$

$$\bar{z}_{bc} = \frac{\bar{z}_{oa}\bar{z}_{ob} + \bar{z}_{ob}\bar{z}_{oc} + \bar{z}_{oc}\bar{z}_{oa}}{\bar{z}_{oa}} \tag{34}$$

$$\bar{z}_{ca} = \frac{\bar{z}_{oa}\bar{z}_{ob} + \bar{z}_{ob}\bar{z}_{oc} + \bar{z}_{oc}\bar{z}_{oa}}{\bar{z}_{ob}} \tag{35}$$

Equations (30) to (35), inclusive, are not limited in their application to three-phase circuits. They give the equivalent star and mesh constants for any three-terminal network.

Equivalent Wye and Delta Impedances for Balanced Loads.—It is often desirable, when solving certain types of problems aris-

ing in engineering, to replace the impedances of a balanced Δ-connected load by impedances connected in wye which take the same power at the same power factor from the three-phase mains. It may also occasionally be desirable to replace a Y-connected load by its equivalent Δ-connected load. In either case, in order to retain the same power factor, the ratio of the resistance to the reactance for the equivalent impedances must be the same as for the impedances they replace. If this ratio is maintained, it is merely necessary, when substituting a Δ-connected load for a Y-connected load, or *vice versa*, to find three impedances which take the same line current at the same line voltages as the original load.

Suppose a balanced Δ-connected load is to be replaced by a balanced Y-connected load. The ratio between the equivalent wye (line) and delta currents of any balanced three-phase system is

$$\frac{I_Y}{I_\Delta} = \sqrt{3}$$

Each of the Y-connected impedances must, therefore, take the square root of three times as much current as each of the equivalent Δ-connected impedances, but the voltages impressed across the Y-connected impedances are only one over the square root of three times as great as the voltage impressed across the Δ-connected impedances. Since the current varies inversely as the impedance of a circuit and directly as the voltage, each of the Y-connected impedances must be

$$\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{1}{3}$$

times as great as each of the Δ-connected impedances, if they are to take the same line current. Conversely, the impedance of each branch of a balanced Δ-connected load must be three times as great as the impedance of each branch of the balance Y-connected load it replaces. This relation between the impedances of equivalent Y- and Δ-connected balanced loads can be obtained from the general equations (30). (31) and (32), pages 312 and 313, by putting $\bar{z}_{ab} = \bar{z}_{bc} = \bar{z}_{ca}$.

Example of the Substitution of a Balanced Y-connected Load for a Balanced Δ-connected Load.—The problem of balanced Y- and Δ-connected loads in parallel, which was solved on page 308, will be solved by replacing the Δ-connected load by its equivalent Y-connected load.

The constants of the actual loads were

$$\bar{z}_Y = 5 + j5$$
$$\bar{z}_\Delta = 10 - j5$$

The impedance of the Y-connected load which will replace the Δ-connected load is

$$\bar{z}_Y{}' = \frac{10}{3} - j\frac{5}{3}$$

The actual wye impedance and the equivalent wye impedance for each phase may be treated as two impedances in parallel across a voltage which is equal to the wye voltage of the system.

The resultant admittance of the two impedances in parallel is

$$\begin{aligned}
\bar{y}_0 = \frac{1}{\bar{z}_Y} + \frac{1}{\bar{z}_Y{}'} &= (g_Y + g_Y{}') - j(b_Y + b_Y{}') \\
&= \left\{\frac{5}{(5)^2 + (5)^2} + \frac{\frac{10}{3}}{\left(\frac{10}{3}\right)^2 + \left(\frac{5}{3}\right)^2}\right\} \\
&\qquad - j\left\{\frac{5}{(5)^2 + (5)^2} + \frac{-\frac{5}{3}}{\left(\frac{10}{3}\right)^2 + \left(\frac{5}{3}\right)^2}\right\} \\
&= (0.1 + 0.24) - j(0.1 - 0.12) \\
&= 0.34 + j0.02
\end{aligned}$$

$$\begin{aligned}
I_{line} = I_0 &= V_{to\ neutral} \times y_0 \\
&= \frac{230}{\sqrt{3}} \times \sqrt{(0.34)^2 + (0.02)^2} \\
&= 132.8 \times 0.3406 \\
&= 45.23 \text{ amperes}
\end{aligned}$$

$$\begin{aligned}
\text{Total power} &= P_0 = 3 \times (V_{to\ neutral})^2 \times g_0 \\
&= 3\left(\frac{230}{\sqrt{3}}\right)^2 \times 0.34 \\
&= 17{,}990 \text{ watts}
\end{aligned}$$

That the substitution of an equivalent Y-connected load for the actual Δ-connected load simplifies the solution of this problem is evident.

Another Example of the Substitution of an Equivalent Y-connected Load for a Balanced Δ-connected Load.—A balanced load, consisting of three equal inductive impedances, each having a resistance of 60 ohms and a reactance of 30 ohms, is connected in delta at the end of a three-phase transmission line which has a resistance of 1 ohm and an inductive reactance of 2 ohms per conductor. If the voltage of the line at the generating station is balanced and is maintained at 2300 volts between

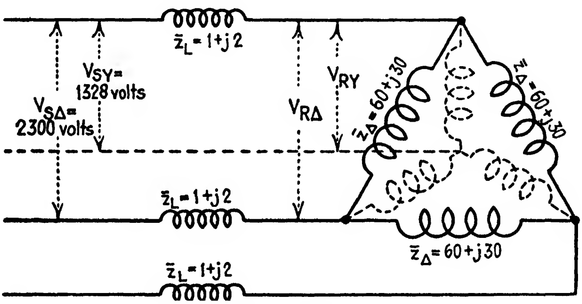


FIG. 100.

conductors, what is the voltage between conductors at the load, *i.e.*, at the receiving end of the line? What is the efficiency of transmission?

This is really a case of a Y-connected load in series with a Δ-connected load, the line impedances being the Y-connected load. The diagram of connections is shown in Fig. 100. The equivalent Y-connected load which replaces the Δ-connected load is shown dotted.

Let V_{SY} and V_{RY} be the voltages to neutral, *i.e.*, the wye voltages, at the station end of the line and at the receiving or load end, respectively. Then,

$$V_{SY} = \frac{2300}{\sqrt{3}} = 1328 \text{ volts}$$

The equivalent Y-connected impedances which will replace the actual Δ-connected impedances are each

$$\bar{z}_{RY} = \frac{\bar{z}_{R\Delta}}{3} = \frac{1}{3}(60 + j30)$$
$$= 20 + j10$$

Since the load is balanced, the neutral points at the load and at the station are at the same potential and no current flows between them even if they are connected. Therefore, the voltage to neutral at the generating end of the line may be assumed to be used up in the potential drop between the station and the neutral point of the equivalent Y-connected load. This drop is equal to the line current multiplied by the resultant impedance of a single conductor and of one phase of the equivalent Y-connected load. Therefore,

$$\bar{I}_{line} = \frac{\bar{V}_{SY}}{\bar{z}_L + \bar{z}_{RY}}$$

where $\bar{z}_L = 1 + j2$ is the impedance of the transmission line per conductor.

$$I_{line} = \frac{1328}{\sqrt{(1 + 20)^2 + (2 + 10)^2}}$$
$$= \frac{1328}{24.19} = 54.89 \text{ amperes}$$

At the load the voltage between line and neutral is

$$V_{RY} = I_{line} z_{RY}$$
$$= 54.89 \times \sqrt{(20)^2 + (10)^2}$$
$$= 1227 \text{ volts}$$

At the load the voltage between lines is

$$V_{R\Delta} = \sqrt{3} \times V_{RY}$$
$$= 1.732 \times 1227$$
$$= 2125 \text{ volts}$$

$$\text{Efficiency of transmission} = \frac{3(I_{line})^2 r_{RY}}{3(I_{line})^2 (r_{RY} + r_L)}$$
$$= \frac{r_{RY}}{r_{RY} + r_L}$$
$$= \frac{20}{20 + 1} = 0.952$$

CHAPTER XI

HARMONICS IN POLYPHASE CIRCUITS

Relative Magnitudes of Line and Phase Currents and of Line and Phase Voltages of Balanced Polyphase Circuits When the Currents and Voltages Are Not Sinusoidal.—Let v_{01}, v_{02} and v_{03} be the instantaneous voltages of a balanced three-phase circuit which contains both even and odd harmonics.

For simplicity, the fundamental and harmonics for each phase are assumed to be zero and increasing in the same direction at the same instant.

$$\begin{aligned} v_{01} = {} & V_{m1} \sin(\omega t) + V_{m2} \sin 2(\omega t) \\ & + V_{m3} \sin 3(\omega t) + V_{m4} \sin 4(\omega t) \\ & + V_{m5} \sin 5(\omega t) + \text{etc.} \end{aligned} \tag{1}$$

$$\begin{aligned} v_{02} = {} & V_{m1} \sin(\omega t - 120°) + V_{m2} \sin 2(\omega t - 120°) \\ & + V_{m3} \sin 3(\omega t - 120°) + V_{m4} \sin 4(\omega t - 120°) \\ & + V_{m5} \sin 5(\omega t - 120°) + \text{etc.} \end{aligned} \tag{2}$$

$$\begin{aligned} v_{03} = {} & V_{m1} \sin(\omega t - 240°) + V_{m2} \sin 2(\omega t - 240°) \\ & + V_{m3} \sin 3(\omega t - 240°) + V_{m4} \sin 4(\omega t - 240°) \\ & + V_{m5} \sin 5(\omega t - 240°) + \text{etc.} \end{aligned} \tag{3}$$

The angular displacement between any harmonic of any phase and the corresponding harmonic of phase 1 is given in the following table.

Phase	Phase displacement in electrical degrees					
	First	Second	Third	Fourth	Fifth	Sixth
1	0°	0°	0°	0°	0°	0°
2	120°	2 × 120° = 240°	3 × 120° = 360° ≎ 0°	4 × 120° = 480° ≎ 120°	5 × 120° = 600° ≎ 240°	6 × 120° = 720° ≎ 0°
3	240°	2 × 240° = 480° ≎ 120°	3 × 240° = 720° ≎ 0°	4 × 240° = 960° ≎ 240°	5 × 240° = 1200° ≎ 120°	6 × 240° = 1440° ≎ 0°

Phase	Phase displacement in electrical degrees				
	Seventh	Eighth	Ninth	Tenth	Eleventh
1	0°	0°	0°	0°	0°
2	7 × 120° = 840° ≈ 120°	8 × 120° = 960° ≈ 240°	9 × 120° = 1080° ≈ 0°	10 × 120° = 1200° ≈ 120°	11 × 120° = 1320° ≈ 240°
3	7 × 240° = 1680° ≈ 240°	8 × 240° = 1920° ≈ 120°	9 × 240° = 2160° ≈ 0°	10 × 240° = 2400° ≈ 240°	11 × 240° = 2640° ≈ 120°

A phase displacement between any two current or voltage vectors of like frequency, of any whole number of complete periods, *i.e.*, of any integral multiple of 360 degrees, does not alter their relative phase.

It is evident from the table that the harmonics of triple frequency are in phase, as are all harmonics which are multiples

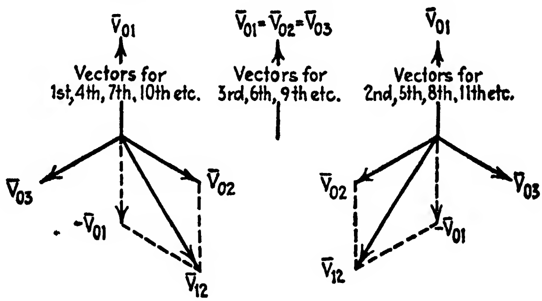


FIG. 101.

of this frequency, such as the sixth, ninth, twelfth, fifteenth etc. If the third harmonics are omitted, as well as all multiples of the third, the phase order of the remaining harmonics, starting with the first, or fundamental, alternates from the order 1, 2, 3 to the order 1, 3, 2. If the fundamental of phase 1 leads the fundamental of phase 2, the second harmonic of phase 1 lags the second harmonic of phase 2. The phase order of the second, fifth, eighth, eleventh etc. harmonics is opposite to that of the fundamental. The phase order of the fourth, seventh, tenth, thirteenth etc. harmonics is the same as that of the fundamental. The vectors for the fundamentals and harmonics are shown in Fig. 101.

From the nature of the construction of alternators, their voltage waves are symmetrical and, therefore, do not contain

even harmonics. For this reason, even harmonics seldom have to be considered when dealing with polyphase circuits, since

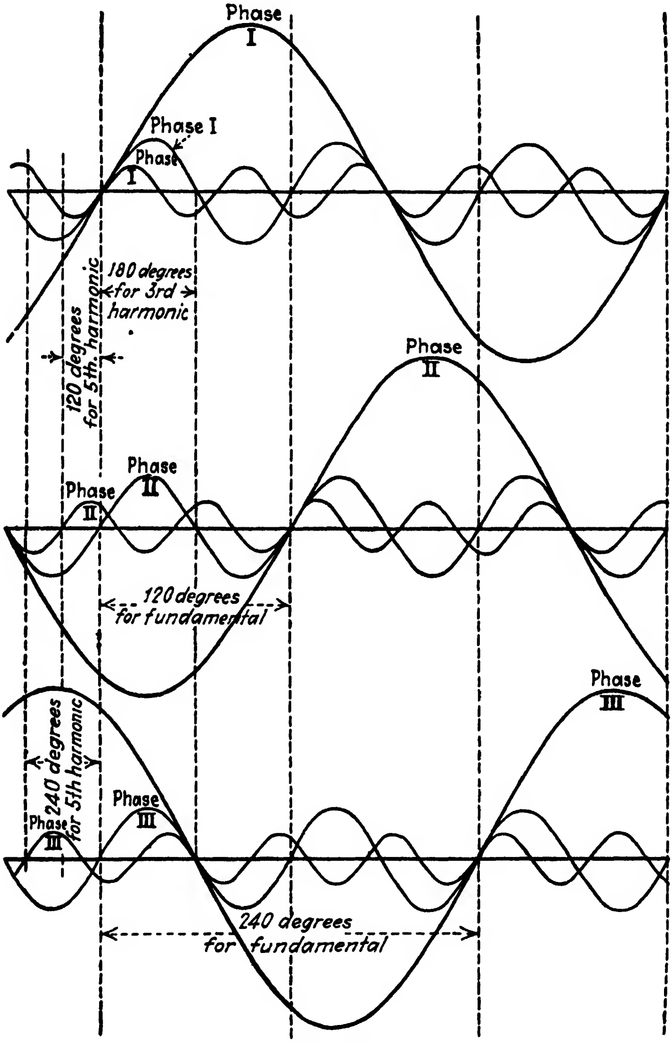


FIG. 102.

such circuits obtain their power from alternators. If third harmonics and their multiples are omitted and even harmonics are neglected, the phase order of the fundamental and the

remaining harmonics alternates, *i.e.*, the phase order of the fundamental and the seventh, thirteenth harmonics etc. is the same as that of the fundamental, and the phase order of the fifth, eleventh harmonics etc. is opposite to it.

The fundamental and the third and fifth harmonics in equations (1), (2) and (3) are plotted in Fig. 102.

Diagrams of wye and delta connections are shown in Fig. 103*a* and *b*.

Wye Connection.—Assume that the phase voltages of a three-phase alternator are given by equations (4), (5) and (6). The

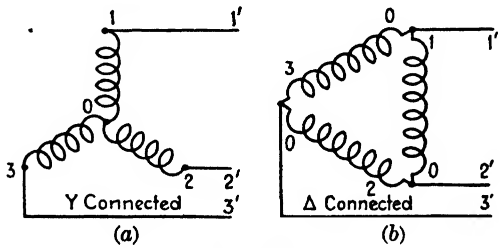


FIG. 103.

fundamentals and the third and fifth harmonics of these voltages are plotted in Fig. 102.

$$v_{01} = V_{m1} \sin (\omega t) + V_{m3} \sin 3(\omega t) + V_{m5} \sin 5(\omega t) + \text{etc.} \tag{4}$$

$$v_{02} = V_{m1} \sin (\omega t - 120°) + V_{m3} \sin 3(\omega t - 120°) + V_{m5} \sin 5(\omega t - 120°) + \text{etc.} \tag{5}$$

$$v_{03} = V_{m1} \sin (\omega t - 240°) + V_{m3} \sin 3(\omega t - 240°) + V_{m5} \sin 5(\omega t - 240°) + \text{etc.} \tag{6}$$

If this alternator is Y-connected, the equations for its terminal voltages are (see Figs. 101 and 103*a*)

$$v_{12} = -v_{01} + v_{02} = \sqrt{3}V_{m1} \sin (\omega t - 150°) + 0 + \sqrt{3}V_{m5} \sin (5\omega t + 150°) + \sqrt{3}V_{m7} \sin (7\omega t - 150°) + \text{etc.} \tag{7}$$

$$v_{23} = -v_{02} + v_{03} = \sqrt{3}V_{m1} \sin (\omega t - 120° - 150°) + 0 + \sqrt{3}V_{m5} \sin (5\omega t + 120° + 150°) + \sqrt{3}V_{m7} \sin (7\omega t - 120° - 150°) + \text{etc.} \tag{8}$$

$$v_{31} = -v_{03} + v_{01} = \sqrt{3}V_{m1} \sin (\omega t - 240° - 150°) + 0$$
$$+ \sqrt{3}V_{m5} \sin (5\omega t + 240° + 150°)$$
$$+ \sqrt{3}V_{m7} \sin (7\omega t - 240° - 150°)$$
$$+ \text{etc.} \tag{9}$$

For equations (4) to (9) inclusive, time t is zero when the fundamental of phase 1 is zero.

It is obvious, from equations (7), (8) and (9), that there can be no harmonic of triple frequency or any harmonic, such as the ninth or fifteenth, whose frequency is a multiple of triple frequency, in the line voltage of a balanced three-phase, Y-connected alternator, even if its phase voltage contains harmonics of triple frequency or multiples of this frequency. The third harmonics in the two phases connected between any pair of line terminals are opposite in phase when considered through the windings from one line terminal to the other and therefore cancel. The same statement is true regarding the ninth harmonic and other harmonics whose frequency is a multiple of triple frequency.

The root-mean-square or effective terminal voltage corresponding to equations (7), (8) and (9) is

$$V_{12} = V_{23} = V_{31}$$
$$= \sqrt{3} \times \sqrt{\frac{V_{m1}^2 + V_{m5}^2 + V_{m7}^2 + V_{m11}^2 + \text{etc.}}{2}} \tag{10}$$

The root-mean-square or effective phase voltage is

$$V_{01} = V_{02} = V_{03}$$
$$= \sqrt{\frac{V_{m1}^2+V_{m3}^2+V_{m5}^2 + V_{m7}^2 + V_{m9}^2 + V_{m11}^2 + \text{etc.}}{2}} \tag{11}$$

Obviously, the ratio of line to phase voltage for a Y-connected alternator cannot be equal to the square root of three when the phase voltages contain harmonics of triple frequency or other harmonics which are multiples of this frequency.

For balanced loads, the same phase relations hold for phase currents as hold for phase voltages. The fundamentals and all harmonics that may exist in the currents, except the harmonics of triple frequency or multiples of this frequency, are 120 degrees apart in phase. Their vector sum is therefore zero, since the vector sum of any three equal vectors which are 120 degrees

apart in phase is zero. The third harmonics, if they exist, are all in phase. Their vector sum therefore is not zero, but is three times the third harmonic for one phase. A similar statement is true of any harmonic whose frequency is a multiple of triple frequency. Since the vector sum of all currents flowing toward any junction point in a circuit must be zero, it is obvious that there cannot be any third-harmonic current, or any current whose frequency is a multiple of triple frequency, in a balanced three-phase, Y-connected alternator under balanced load conditions, unless the neutral point of the load and the neutral point of the alternator are interconnected. Without the neutral, the only way the vector sum of the harmonics of triple frequency and multiples of this frequency can be zero at the neutral point is for the harmonics themselves to be zero.

When the neutrals are interconnected, the neutral conductor acts as a common return for any third-harmonic phase currents that may exist and carries a third-harmonic current equal to three times the third harmonic current per phase, a balanced load being assumed. It also acts as a common return for any other component phase currents whose frequency is a multiple of the triple frequency. It carries three times the ninth harmonic phase current, if any exists, and three times any other multiple of the third harmonic current which is present in the phases.

So long as the load remains balanced, the neutral conductor carries no fundamental current and no fifth, seventh, eleventh etc. harmonic currents. The three-phase currents in each of these groups are equal and 120 degrees apart for balanced conditions, and their sum is zero at the neutral connection.

The neutral connection between the load and the alternator supplying it may be considered to carry component currents each equal to the vector sum of the corresponding component currents in the phases. Under balanced conditions the vector sums of all component currents in the phases of like frequencies except those of triple frequencies are zero. When the load is unbalanced, these vector sums are not zero. Under this condition the neutral connection may carry component currents of any frequency.

The fact that wye connection for balanced loads, without neutral connection between the load and the source of power, suppresses any third harmonic current which might otherwise

exist is of considerable importance under certain conditions. When a third-harmonic current is necessary for the successful operation of a system, a wye connection, which has no neutral conductor connecting the neutral of the load to the neutral of the source of power, cannot be used. In other cases, where it is desirable to suppress a third harmonic current, wye connection is highly desirable.

Delta Connection.—There can be no third harmonic or multiples of the third harmonic in the terminal voltage of a Δ-connected alternator. In case any third harmonics exist in the phase voltages of a Δ-connected alternator, they are in phase around the closed delta formed by the armature windings and are therefore short-circuited. The same statement is true in regard to the ninth harmonic current and any other multiple of the third-harmonic current.

If E_3 is the effective value of the third-harmonic voltage in each phase of a three-phase, Δ-connected alternator and z_3 is the phase impedance for the third-harmonic frequency, the effective value, I_3, of the third-harmonic current short-circuited in the closed delta is

$$I_3 = \frac{3E_3}{3z_3}$$

$$I_3 z_3 = E_3$$

The third-harmonic voltage in each phase is used up in the third-harmonic impedance drop in that phase and cannot appear, therefore, at the terminals of the alternator. The possibility of a short-circuit current of triple frequency in the armature of a Δ-connected alternator is one reason why delta connection for alternators is less satisfactory than wye connection.

Although there can be no harmonic of triple frequency, or any multiple of this frequency, in the terminal voltage of either a Y- or Δ-connected, three-phase alternator, there may be a harmonic of triple frequency and harmonics whose frequencies are multiples of this frequency between the line terminals and neutral connection of a Y-connected alternator.

There can be no harmonic currents of triple frequency or multiples of this frequency in the conductors connecting a balanced Δ-connected load to a balanced Δ-connected alternator,

since those harmonics, even if they existed in the load, would be in phase around the delta and their sum flowing toward any junction point between a line and two phases would therefore be zero. Any third harmonics, ninth harmonics or other multiples of the third harmonic that existed in the load would merely circulate around the closed delta of the load, without appearing on the lines.

The line and phase currents in each phase of a balanced Y-connected alternator or load are obviously the same both in magnitude and in phase. For a Δ-connected alternator, the current in any line is the vector sum of the phase currents in the two adjoining phases, both phase currents being considered toward the junction point. Referring to Fig. 103*b*, page 321, the current in line 1 is

$$\begin{aligned} \bar{I}_{1'1} &= \bar{I}_{10} + \bar{I}_{03} \\ &= -\bar{I}_{01} + \bar{I}_{03} \end{aligned}$$

The phase relations given in the table on pages 318 and 319 for the harmonics in the phase voltages of a balanced three-phase alternator also hold for the phase relations of the phase currents in any balanced three-phase alternator or load. In practice, many of the harmonics may be missing or too small in magnitude to be considered. In certain cases, however, some of the harmonics may be large. This is especially true for inductive loads containing iron which is worked at high saturation. The current taken by such a circuit is non-sinusoidal and contains a very marked third harmonic, even if the impressed voltage is sinusoidal. (See page 251.)

If the phases marked 01, 02 and 03, Fig. 103*b*, page 321, correspond to phases 1, 2 and 3, respectively, in the table, the instantaneous line currents $i_{1'1}$, $i_{2'2}$ and $i_{3'3}$ of a Δ-connected alternator, which contains odd harmonics of all orders in its phase voltages, is (i_{02} assumed lagging i_{01})

$$\begin{aligned} i_{1'1} = -i_{01} + i_{03} = \sqrt{3}I_{m1} &\sin(\omega t - \theta_1 + 150°) + 0 \\ &+ \sqrt{3}I_{m5} \sin(5\omega t - \theta_5 - 150°) \\ &+ \sqrt{3}I_{m7} \sin(7\omega t - \theta_7 + 150°) \\ &+ \text{etc.} \end{aligned} \tag{12}$$

$$
\begin{aligned}
i_{2'2} = -i_{02} + i_{01} = \sqrt{3}I_{m1} &\sin(\omega t - 120° - \theta_1 + 150°) + 0 \\
&+ \sqrt{3}I_{m5} \sin(5\omega t + 120° - \theta_5 - 150°) \\
&\quad + \sqrt{3}I_{m7} \sin(7\omega t - 120 - \theta_7 + 150°) \\
&\qquad + \text{etc.} \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
i_{3'3} = -i_{03} + i_{02} = \sqrt{3}I_{m1} &\sin(\omega t - 240° - \theta_1 + 150°) + 0 \\
&+ \sqrt{3}I_{m5} \sin(5\omega t + 240° - \theta_5 - 150°) \\
&\quad + \sqrt{3}I_{m7} \sin(7\omega t - 240° - \theta_7 + 150°) \\
&\qquad + \text{etc.} \qquad (14)
\end{aligned}
$$

where the I_m's with subscripts 1, 3, 5 etc. represent the maximum values of the fundamental and the harmonics. The angles θ are the angles of lag between the component currents and the corresponding component voltages. The angles θ are different for the fundamental and the harmonics. For constant inductance, constant capacitance and constant resistance in series, the angle of lag for any harmonic, such as the nth, is

$$\theta_n = \tan^{-1}\frac{nx_L + \frac{x_C}{n}}{r}$$

where $x_L = \omega L$ and $x_C = \frac{-1}{\omega C}$ are the inductive and capacitive reactances for the fundamental frequency.

Equivalent Wye and Delta Voltages of Balanced Three-phase Systems Which Have Non-sinusoidal Waves and Which Contain Only Odd Harmonics.—In general, there can be no third-harmonic voltage between the lines of any balanced three-phase circuit. If there can be no third harmonic in the voltage between the lines of a balanced three-phase system, obviously there can be no third harmonic in the equivalent wye voltage of such a system. The equivalent wye voltage is therefore equal to the line voltage divided by the square root of three. When there is no harmonic of triple frequency or any multiple of this frequency, it is evident from equations (10) and (11), page 322, that the ratio of line to phase voltage is equal to the square root of three for wye connection. This relation between the line voltage and the equivalent wye voltage of a balanced three-phase circuit is used in determining the power factor of a balanced three-phase system which may contain harmonics.

An Example Illustrating the Relations between Line and Phase Voltages of a Three-phase Alternator Which May Be Connected either in Wye or in Delta.—Oscillograph records show that the phase voltage of a certain sixty-cycle, three-phase, Y-connected alternator contains third, fifth and seventh harmonics but no harmonics of higher order of appreciable magnitude and no even harmonics. The expression for the phase voltage is

$$e = 1880 \sin 377t + 175 \sin (1131t - 25°) + 75 \sin (1885t - 30°) + 30 \sin (2639t + 40°)$$

If time is reckoned from the instant when the fundamental of phase 1 is passing through zero increasing in a positive direction, what are the expressions for the voltages of the three phases? What are the expressions for the three line voltages? What are the root-mean-square values of the line and phase voltages for wye connection? What is their ratio? If the alternator is reconnected in delta, what are the three line voltages? What is the ratio of the line voltages for wye and for delta connection?

The phase voltages are (see table on page 318)

$$\begin{aligned}
e_{01} &= 1880 \sin 377t + 175 \sin (1131t - 25°) \\
&\quad + 75 \sin (1885t - 30°) \\
&\qquad + 30 \sin (2639t + 40°) \\
e_{02} &= 1880 \sin (377t - 120°) + 175 \sin (1131t - 25° \pm 0°) \\
&\quad + 75 \sin (1885t - 30° + 120°) \\
&\qquad + 30 \sin (2639t + 40° - 120°) \\
&= 1880 \sin (377t - 120°) + 175 \sin (1131t - 25° \pm 0°) \\
&\quad + 75 \sin (1885t + 90°) \\
&\qquad + 30 \sin (2639t - 80°) \\
e_{03} &= 1880 \sin (377t - 240°) + 175 \sin (1131t - 25° \pm 0°) \\
&\quad + 75 \sin (1885t - 30° + 240°) \\
&\qquad + 30 \sin (2639t + 40° - 240°) \\
&= 1880 \sin (377t - 240°) + 175 \sin (1131t - 25°) \\
&\quad + 75 \sin (1885t + 210°) \\
&\qquad + 30 \sin (2639t - 200°)
\end{aligned}$$

For wye connection (see Figs. 101 and 103*a*, pages 319 and 321),

$$
\begin{aligned}
e_{12} &= -e_{01} + e_{02} \\
&= \sqrt{3} \times 1880 \sin (377t - 150°) + 0 \\
&\quad + \sqrt{3} \times 75 \sin (1885t - 30° + 150°) \\
&\qquad + \sqrt{3} \times 30 \sin (2639t + 40° - 150°) \\
&= 3256 \sin (377t - 150°) \\
&\quad + 130 \sin (1885t + 120°) \\
&\qquad + 52 \sin (2639t - 110°) \\
e_{23} &= -e_{02} + e_{03} \\
&= \sqrt{3} \times 1880 \sin (377t - 120° - 150°) + 0 \\
&\quad + \sqrt{3} \times 75 \sin (1885t - 30° + 120° + 150°) \\
&\qquad + \sqrt{3} \times 30 \sin (2639t + 40° - 120° - 150°) \\
&= 3256 \sin (377t + 90°) \\
&\quad + 130 \sin (1885t - 120°) \\
&\qquad + 52 \sin (2639t + 130°) \\
e_{31} &= -e_{03} + e_{01} \\
&= \sqrt{3} \times 1880 \sin (377t - 240° - 150°) + 0 \\
&\quad + \sqrt{3} \times 75 \sin (1885t - 30° + 240° + 150°) \\
&\qquad + \sqrt{3} \times 30 \sin (2639t + 40° - 240° - 150°) \\
&= 3256 \sin (377t - 30°) \\
&\quad + 130 \sin (1885t + 0°) \\
&\qquad + 52 \sin (2639t + 10°)
\end{aligned}
$$

The root-mean-square phase voltage is

$$
\begin{aligned}
V_{phase}\ (\text{wye}) &= \sqrt{\frac{(1880)^2 + (175)^2 + (75)^2 + (30)^2}{2}} \\
&= 1336 \text{ volts}
\end{aligned}
$$

The root-mean-square line voltage for wye connection is

$$
\begin{aligned}
V_{line}\ (\text{wye}) &= \sqrt{\frac{(3256)^2 + (130)^2 + (52)^2}{2}} \\
&= 2305 \text{ volts}
\end{aligned}
$$

The ratio of the line and phase voltages for wye connection is

$$
\frac{V_{line}\ (\text{wye connection})}{V_{phase}\ (\text{wye connection})} = \frac{2305}{1336} = 1.725
$$

This ratio would be exactly equal to the square root of three if there were no third harmonic in the phase voltage.

For delta connection, the third-harmonic phase voltage is short-circuited in the closed circuit formed by the Δ-connected

armature winding and therefore cannot appear between the line terminals. If the delta connection is made as shown in Fig. 103*b*, page 321, the line voltages are

$$\begin{aligned} e_{21} &= 1880 \sin 377t + 75 \sin (1885t - 30°) \\ &\quad +30 \sin (2639t + 40°) \\ e_{13} &= 1880 \sin (377t - 240°) + 75 \sin (1885t + 210°) \\ &\quad + 30 \sin (2639t - 200°) \\ e_{32} &= 1880 \sin (377t - 120°) + 75 \sin (1885t + 90°) \\ &\quad + 30 \sin (2639t - 80°) \end{aligned}$$

The root-mean-square line voltage for delta connection is

$$\begin{aligned} V_{line} \text{ (delta)} &= \sqrt{\frac{(1880)^2 + (75)^2 + (30)^2}{2}} \\ &= 1331 \end{aligned}$$

The ratio of the root-mean-square line voltages for wye and delta connections is

$$\frac{V_{line} \text{ (wye connection)}}{V_{line} \text{ (delta connection)}} = \frac{2305}{1331} = 1.732 = \sqrt{3}$$

This ratio is equal to the square root of three, as it must be, since neither voltage contains the third harmonic.

An Example Involving Harmonics in a Balanced Three-phase Circuit.—Three identical non-inductive resistances, when connected in delta across the terminals of a balanced Y-connected alternator, consume 12,000 watts. When these same resistances are connected in wye across the same alternator, they consume 4750 watts when their common connection, *i.e.*, their neutral point, is connected to the neutral of the alternator. Under this condition the current in the neutral is 15 amperes. The voltage of the alternator is assumed to be the same in both cases. What are the root-mean-square values of the voltages between the line terminals of the alternator and between its line terminals and neutral point?

If there were no harmonics of triple frequency or multiples of this frequency present in the voltage of the alternator, the power consumed by the resistances when connected in wye would be equal to one-third of the power they consume when connected in

delta across the same voltage. This follows from the fact that power in a circuit is

$$P = V^2 g = V^2 \frac{r}{r^2 + x^2}$$

When there is only pure resistance, *i.e.*, when the inductive and capacitive reactances for all frequencies are zero, the expression for power reduces to the voltage squared divided by the resistance. The expression

$$P = V^2 g$$

applies to one particular harmonic, that for which the conductance g is computed. The conductance has a different value for the fundamental and each harmonic, except when the inductive and capacitive reactances for all frequencies are zero and the resistance is constant. If the inductive and capacitive reactances for all frequencies are zero and the resistance is assumed to be constant, the expression

$$P = \frac{V^2}{r}$$

for the power consumed by a circuit which has no reactance holds for all harmonics and the fundamental, provided there are no harmonics in the voltage which are not present in the current.

$$\frac{\text{Power in delta}}{\text{Power in wye}} = \frac{(\text{delta voltage})^2}{(\text{wye voltage})^2}$$

If there were no third harmonic or multiples of the third present in the voltage, the ratio of delta voltage squared to wye voltage squared would be $(\sqrt{3})^2 = 3$.

Therefore, if no harmonics of triple frequency or multiples of this frequency were present in the current when the resistances were connected in wye, they would consume

$$\frac{12{,}000}{3} = 4000 \text{ watts}$$

The difference between the actual power consumed by the three resistances when Y-connected, with their neutral point and the neutral point of the alternator interconnected, and 4000 watts must be due to harmonic currents of triple frequency or multiples of this frequency which return on the neutral.

The harmonic currents of triple frequency and multiples of this frequency in each of the three resistances when Y-connected must be equal to one-third the neutral current.

$$\frac{15}{3} = 5 \text{ amperes}$$

The copper loss, *i.e.*, the I^2r loss, due to harmonic currents of triple frequency and multiples of this frequency in each of the resistances when Y-connected, is

$$\frac{4750 - 4000}{3} = \frac{750}{3} = 250 \text{ watts}$$

Therefore,

$$(5)^2 \times r = 250$$
$$r = 10 \text{ ohms per resistance unit}$$

When the resistances are Δ-connected, the power consumed, which is 12,000 watts, or 4000 watts per resistance unit, must be due to a current that does not contain any harmonics of triple frequency or multiples of this frequency. This current must be

$$I = \sqrt{\frac{4000}{10}} = 20 \text{ amperes}$$

Therefore,

$$V_\Delta = 20 \times 10 = 200 \text{ volts}$$
$$V_Y = \sqrt{\left(\frac{200}{\sqrt{3}}\right)^2 + (5 \times 10)^2}$$
$$= 125.8 \text{ volts}$$

Harmonics in Balanced Four-phase Circuits.—Let v_{01}, v_{02}, v_{03} and v_{04} be the instantaneous voltages of a balanced four-phase circuit which contains both even and odd harmonics. For simplicity, the fundamental and harmonics for each phase are assumed to be zero and increasing in the same direction at the same instant.

$$\begin{aligned} v_{01} = V_{m1} \sin \omega t &+ V_{m2} \sin 2\omega t \\ &+ V_{m3} \sin 3\omega t + V_{m4} \sin 4\omega t \\ &+ V_{m5} \sin 5\omega t + V_{m6} \sin 6\omega t \\ &+ V_{m7} \sin 7\omega t + \text{etc.} \end{aligned} \qquad (15)$$

$$\begin{aligned} v_{02} = {} & V_{m1} \sin (\omega t - 90°) + V_{m2} \sin 2(\omega t - 90°) \\ & + V_{m3} \sin 3(\omega t - 90°) + V_{m4} \sin 4(\omega t - 90°) \\ & + V_{m5} \sin 5(\omega t - 90°) + V_{m6} \sin 6(\omega t - 90°) \\ & + V_{m7} \sin 7(\omega t - 90°) + \text{etc.} \end{aligned} \tag{16}$$

$$\begin{aligned} v_{03} = {} & V_{m1} \sin (\omega t - 180°) + V_{m2} \sin 2(\omega t - 180°) \\ & + V_{m3} \sin 3(\omega t - 180°) + V_{m4} \sin 4(\omega t - 180°) \\ & + V_{m5} \sin 5(\omega t - 180°) + V_{m6} \sin 6(\omega t - 180°) \\ & + V_{m7} \sin 7(\omega t - 180°) + \text{etc.} \end{aligned} \tag{17}$$

$$\begin{aligned} v_{04} = {} & V_{m1} \sin (\omega t - 270°) + V_{m2} \sin 2(\omega t - 270°) \\ & + V_{m3} \sin 3(\omega t - 270°) + V_{m4} \sin 4(\omega t - 270°) \\ & + V_{m5} \sin 5(\omega t - 270°) + V_{m6} \sin 6(\omega t - 270°) \\ & + V_{m7} \sin 7(\omega t - 270°) + \text{etc.} \end{aligned} \tag{18}$$

The angular displacement between any harmonic in any phase and the corresponding harmonic in phase 1 is given in the following table.

Phase	Phase displacement in electrical degrees					
	First	Second	Third	Fourth	Fifth	Sixth
1	0°	0°	0°	0°	0°	0°
2	90°	$2 \times 90° = 180°$	$3 \times 90° = 270°$	$4 \times 90° = 360° \Leftrightarrow 0°$	$5 \times 90° = 450° \Leftrightarrow 90°$	$6 \times 90° = 540° \Leftrightarrow 180°$
3	180°	$2 \times 180° = 360° \Leftrightarrow 0°$	$3 \times 180° = 540° \Leftrightarrow 180°$	$4 \times 180° = 720° \Leftrightarrow 0°$	$5 \times 180° = 900° \Leftrightarrow 180°$	$6 \times 180° = 1080° \Leftrightarrow 0°$
4	270°	$2 \times 270° = 540° \Leftrightarrow 180°$	$3 \times 270° = 810° \Leftrightarrow 90°$	$4 \times 270° = 1080° \Leftrightarrow 0°$	$5 \times 270° = 1350° \Leftrightarrow 270°$	$6 \times 270° = 1620° \Leftrightarrow 180°$

	Seventh	Eighth	Ninth	Tenth	Eleventh
1	0°	0°	0°	0°	0°
2	$7 \times 90° = 630° \Leftrightarrow 270°$	$8 \times 90° = 720° \Leftrightarrow 0°$	$9 \times 90° = 810° \Leftrightarrow 90°$	$10 \times 90° = 900° \Leftrightarrow 180°$	$11 \times 90° = 990° \Leftrightarrow 270°$
3	$7 \times 180° = 1260° \Leftrightarrow 180°$	$8 \times 180° = 1440° \Leftrightarrow 0°$	$9 \times 180° = 1620° \Leftrightarrow 180°$	$10 \times 180° = 1800° \Leftrightarrow 0°$	$11 \times 180° = 1980° \Leftrightarrow 180°$
4	$7 \times 270° = 1890° \Leftrightarrow 90°$	$8 \times 270° = 2160° \Leftrightarrow 0°$	$9 \times 270° = 2430° \Leftrightarrow 270°$	$10 \times 270° = 2700° \Leftrightarrow 180°$	$11 \times 270° = 2970° \Leftrightarrow 90°$

It is evident from the table that the harmonics of quadruple frequency are in phase, as are all harmonics which are multiples of this frequency. The harmonics of double frequency, taken in the order of the phases, are successively 180 degrees out of phase. This is also true of all odd multiples of the second harmonic, *i.e.*, the sixth, the tenth, the fourteenth etc. If the second harmonic and its odd multiples and the fourth harmonic and all of its multiples are omitted, the remaining harmonics starting with the fundamental alternate in phase order.

The vectors V_{01}, V_{02}, V_{03} and V_{04}, for the fundamentals and harmonics for a balanced four-phase circuit or load, are shown in Fig. 104.

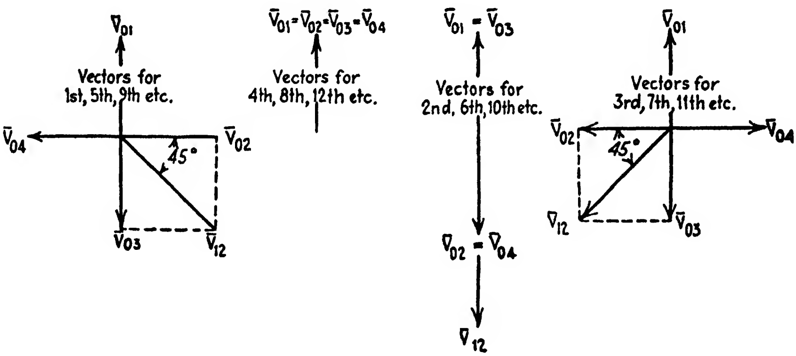


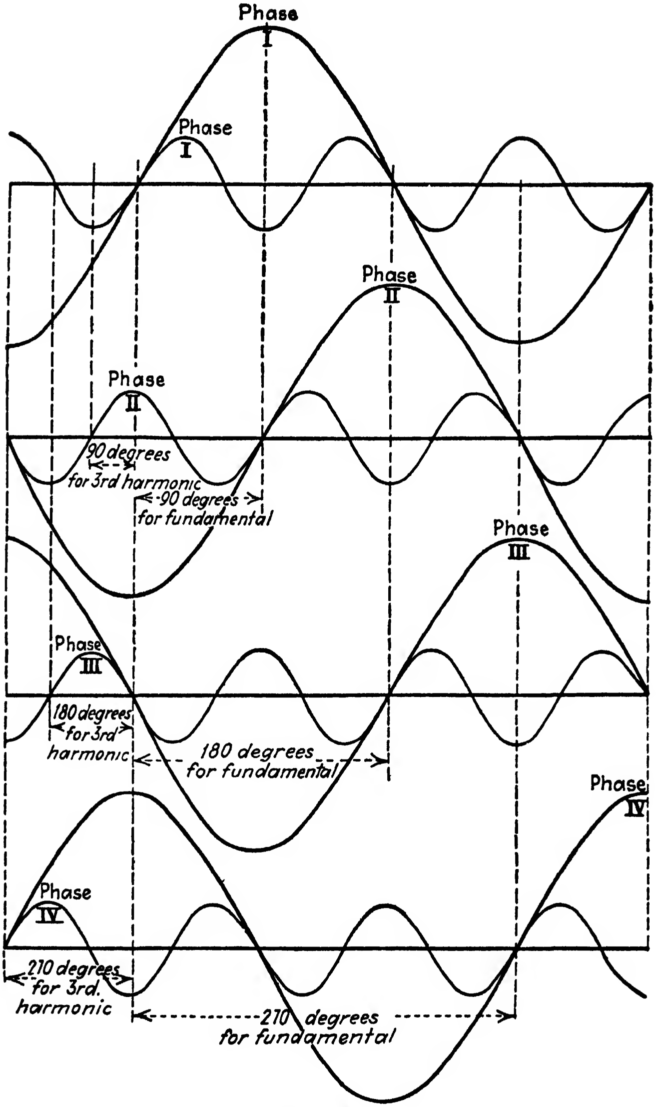
Fig. 104.

If the even harmonics are omitted, since they do not exist in the voltages of alternators, the phase order of the fundamentals and the remaining harmonics alternates, *i.e.*, the phase order of the fifth, ninth etc. harmonics is the same as that of the fundamental, and the phase order of the third, seventh, eleventh etc. is opposite to that of the fundamental.

The fundamentals and third harmonics in equations (15), (16), (17) and (18) are plotted in Fig. 105.

Star and mesh four-phase connections are shown diagrammatically in Fig. 106.

Assume that the phase voltages of a four-phase alternator are given by equations (19), (20), (21) and (22). Even harmonics cannot exist in the voltages of an alternator. (See page 319.)



Phase I
Phase I
Phase II
Phase II
90 degrees
for 3rd harmonic
90 degrees
for fundamental
Phase III
Phase III
180 degrees
for 3rd
harmonic
180 degrees
for fundamental
Phase IV
Phase IV
210 degrees
for 3rd.
harmonic
210 degrees
for fundamental

Fig. 105.

The fundamentals and third harmonics of these voltages are plotted in Fig. 105.

$$\begin{aligned} v_{01} = {} & V_{m1} \sin \omega t + V_{m3} \sin 3\omega t \\ & + V_{m5} \sin 5\omega t + V_{m7} \sin 7\omega t \\ & + \text{etc.} \end{aligned} \tag{19}$$

$$\begin{aligned} v_{02} = {} & V_{m1} \sin (\omega t - 90°) + V_{m3} \sin 3(\omega t - 90°) \\ & + V_{m5} \sin 5(\omega t - 90°) + V_{m7} \sin 7(\omega t - 90°) \\ & + \text{etc.} \end{aligned} \tag{20}$$

$$\begin{aligned} v_{03} = {} & V_{m1} \sin (\omega t - 180°) + V_{m3} \sin 3(\omega t - 180°) \\ & + V_{m5} \sin 5(\omega t - 180°) + V_{m7} \sin 7(\omega t - 180°) \\ & + \text{etc.} \end{aligned} \tag{21}$$

$$\begin{aligned} v_{04} = {} & V_{m1} \sin (\omega t - 270°) + V_{m3} \sin 3 (\omega t - 270°) \\ & + V_{m5} \sin 5(\omega t - 270°) + V_{m7} \sin 7(\omega t - 270°) \\ & + \text{etc.} \end{aligned} \tag{22}$$

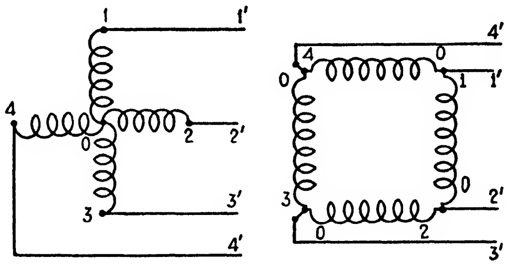


FIG. 106.

The line voltages for the fundamental and harmonics, for star connection, are

$$\bar{V}_{12} = -\bar{V}_{01} + \bar{V}_{02} \tag{23}$$
$$\bar{V}_{23} = -\bar{V}_{02} + \bar{V}_{03} \tag{24}$$
$$\bar{V}_{34} = -\bar{V}_{03} + \bar{V}_{04} \tag{25}$$
$$\bar{V}_{41} = -\bar{V}_{04} + \bar{V}_{01} \tag{26}$$

Since the component voltages of like frequency in the four phases, in equations (23), (24), (25) and (26), differ in time phase by 90 degrees, the line voltage for the fundamental and the line voltage for any harmonic is

$$\begin{aligned} V_{line} &= 2V_{phase} \cos 45° \\ &= \sqrt{2}\, V_{phase} \end{aligned} \tag{27}$$

where V_{line} is the magnitude of the voltage between any pair of adjacent lines for the fundamental or for the harmonics and V_{phase} is the magnitude of the corresponding voltage in the phases.

The instantaneous voltages between lines, *i.e.*, line voltages, for star connection, corresponding to the phase voltages given in equations (19), (20), (21) and (22) are (see Figs. 104 and 106)

$$\begin{aligned} v_{12} = -v_{01} + v_{02} = \sqrt{2}V_{m1} \sin (\omega t - 135°) \\ + \sqrt{2}V_{m3} \sin (3\omega t + 135°) \\ + \sqrt{2}V_{m5} \sin (5\omega t - 135°) \\ + \sqrt{2}V_{m7} \sin (7\omega t + 135°) \\ + \text{etc.} \end{aligned} \tag{28}$$

$$\begin{aligned} v_{23} = -v_{02} + v_{03} = \sqrt{2}V_{m1} \sin (\omega t - 90° - 135°) \\ + \sqrt{2}V_{m3} \sin (3\omega t + 90° + 135°) \\ + \sqrt{2}V_{m5} \sin (5\omega t - 90° - 135°) \\ + \sqrt{2}V_{m7} \sin (7\omega t + 90° + 135°) \\ + \text{etc.} \end{aligned} \tag{29}$$

$$\begin{aligned} v_{34} = -v_{03} + v_{04} = \sqrt{2}V_{m1} \sin (\omega t - 180° - 135°) \\ + \sqrt{2}V_{m3} \sin (3\omega t + 180° + 135°) \\ + \sqrt{2}V_{m5} \sin (5\omega t - 180° - 135°) \\ + \sqrt{2}V_{m7} \sin (7\omega t + 180° + 135°) \\ + \text{etc.} \end{aligned} \tag{30}$$

$$\begin{aligned} v_{41} = -v_{04} + v_{01} = \sqrt{2}V_{m1} \sin (\omega t - 270° - 135°) \\ + \sqrt{2}V_{m3} \sin (3\omega t + 270° + 135°) \\ + \sqrt{2}V_{m5} \sin (5\omega t - 270° - 135°) \\ + \sqrt{2}V_{m7} \sin (7\omega t + 270° + 135°) \\ + \text{etc.} \end{aligned} \tag{31}$$

The root-mean-square values of the line and phase voltages for star connection are

$$V_{phase} = \sqrt{\frac{V_{m1}^2 + V_{m3}^2 + V_{m5}^2 + V_{m7}^2 + \text{etc.}}{2}} \tag{32}$$

$$\begin{aligned} V_{line} &= \sqrt{2}\, V_{phase} \\ &= \sqrt{2} \times \sqrt{\frac{V_{m1}^2 + V_{m3}^2 + V_{m5}^2 + V_{m7}^2 + \text{etc.}}{2}} \end{aligned} \tag{33}$$

$$\frac{V_{line}}{V_{phase}} = \sqrt{2} \tag{34}$$

The ratio of line to phase root-mean-square voltage for a balanced four-phase, star-connected alternator is always equal to the square root of two, since even harmonics are not present in the voltages of alternators. This is entirely independent of wave form. Although the line and phase voltages contain like harmonics in the same relative magnitudes, the wave forms of the two voltages are not alike. The relative phase displacements of the harmonics in the two voltages are different, as is evident by comparing equations (28), (29), (30) and (31) on page 336 with equations (19), (20), (21) and (22) on page 335.

The line and phase voltages of a mesh-connected, four-phase alternator are the same in both magnitude and wave form, since no harmonic is short-circuited in a mesh-connected, four-phase circuit and thus eliminated from the terminal voltage. This statement assumes that even harmonics are not present.

For balanced conditions, the same relations hold between line and phase *currents* for four-phase *mesh connection* as hold between line and phase *voltages* for a balanced four-phase, *star-connected* circuit.

If the neutral of a balanced four-phase, star-connected load is connected to the neutral of the source of power supplying the load, no current flows in the neutral, since the fundamentals of the phase currents and all harmonics of any order that may exist in the phase currents are 90 degrees apart in time phase and therefore their sum is zero at the neutral point. The vector sum of any four equal vectors which differ in phase by 90 degrees is equal to zero. The only current carried by the neutral connection of a four-phase, star-connected system is that due to an unbalanced load. This statement assumes that only odd harmonics are present. If fourth-harmonic currents and their multiples existed in the phases, they would be in phase and would add directly on the neutral like the third-harmonic currents of a Y-connected, three-phase alternator.

Since the harmonics as well as the fundamentals in the phase voltages of a balanced four-phase alternator are 90 degrees apart in time phase, there can be no circulatory current in the armature of a mesh-connected, four-phase alternator, whose voltages are balanced, since the vector sum of the component voltages of any given frequency is zero. This is different

from the conditions which may exist in the armature of a three-phase, mesh-connected, *i.e.*, Δ-connected alternator. In this case, the third harmonics and all harmonics whose frequency is a multiple of triple frequency are short-circuited in the closed delta formed by the armature windings. If fourth harmonics or their multiples existed in the phase voltages of a four-phase, mesh-connected alternator, they would be short-circuited like the third harmonics and their multiples in a three-phase, Δ-connected alternator.

There can be no harmonic of triple frequency or any multiple of this frequency in the line voltage of a balanced three-phase alternator or circuit. With a four-phase alternator or circuit, the third harmonic and other harmonics whose frequencies are multiples of triple frequency are not cut out from the line voltages. Odd harmonics of any order may appear in both the line and the phase voltages for four-phase connection. None is suppressed by this connection.

Harmonics in Balanced Six-phase Circuits.—Although no six-phase alternators are built, there is a demand for a considerable amount of six-phase power for the operation of synchronous converters and mercury-arc rectifiers. This is always obtained from three-phase systems by means of ordinary static transformers connected for three-phase to six-phase transformation. Alternating-current generation and transmission of power are universally used. Direct-current distribution is frequently employed in the business and thickly settled districts of many of the older cities. The synchronous converter or mercury-arc rectifier is also used in connection with most street railways operating on direct current. Synchronous converters usually operate from six-phase circuits. Mercury-arc rectifiers for heavy power work operate on either six-phase or twelve-phase circuits. The larger number of phases gives a smoother rectified wave.

On account of the importance of the synchronous converter and the mercury-arc rectifier as connecting links between alternating-current transmission and direct-current distribution, it is worth while to give some consideration to the relations among the harmonics in six-phase circuits.

Six-phase systems are always derived from polyphase systems, usually three-phase, by means of transformers. A six-phase

system may be considered to be two three-phase systems which are superposed, with the voltages of one reversed with respect to those of the other. This reversal may be obtained by merely reversing the connections of the secondary windings of the transformers supplying one of the superposed systems. Consider

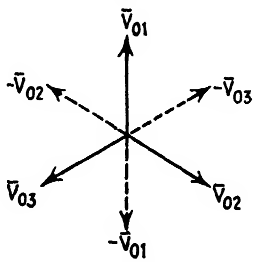


FIG. 107.

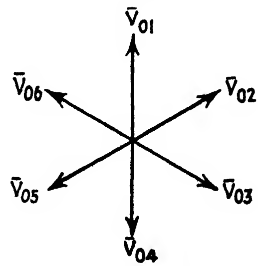


FIG. 108.

Fig. 107. This figure shows the vectors representing the fundamental voltages of a six-phase system.

The two superposed three-phase systems, to which the six-phase system is equivalent, are distinguished in Fig. 107 by using full lines for one and dotted lines for the other.

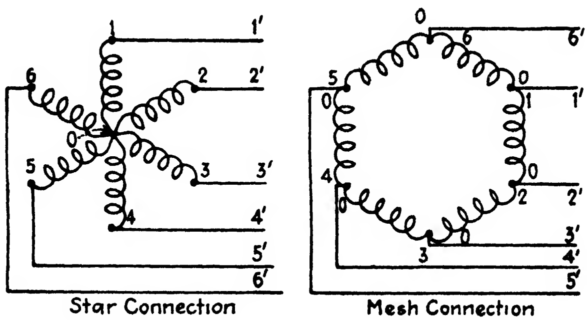


FIG. 109.

Figure 108 shows the six voltage vectors for the fundamentals of a six-phase system.

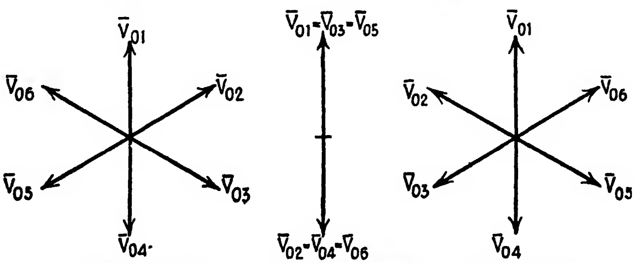
Either star or mesh connection may be used for six-phase circuits. Star and mesh connections are illustrated diagrammatically in Fig. 109.

The following table gives the angular displacement between any harmonic in any phase and the corresponding harmonic in

phase 1. Only odd harmonics are considered in this table. The phase relations between the even harmonics may readily be supplied when needed.

Phase	Phase displacement in electrical degrees					
	1st	3d	5th	7th	9th	11th
1	0°	0°	0°	0°	0°	0°
2	60°	3 × 60° = 180°	5 × 60° = 300°	7 × 60° = 420° ≎ 60°	9 × 60° = 540° ≎ 180°	11 × 60° = 660° ≎ 300°
3	120°	3 × 120° = 360° ≎ 0°	5 × 120° = 600° ≎ 240°	7 × 120° = 840° ≎ 120°	9 × 120° = 1080° ≎ 0°	11 × 120° = 1320° ≎ 240°
4	180°	3 × 180° = 540° ≎ 180°	5 × 180° = 900° ≎ 180°	7 × 180° = 1260° ≎ 180°	9 × 180° = 1620° ≎ 180°	11 × 180° = 1980° ≎ 180°
5	240°	3 × 240° = 720° ≎ 0°	5 × 240° = 1200° ≎ 120°	7 × 240° = 1680° ≎ 240°	9 × 240° = 2160° ≎ 0°	11 × 240° = 2640° ≎ 120°
6	300°	3 × 300° = 900° ≎ 180°	5 × 300° = 1500° ≎ 60°	7 × 300° = 2100° ≎ 300°	9 × 300° = 2700° ≎ 180°	11 × 300° = 3300° ≎ 60°

The vectors for the fundamental and for the odd harmonics of a balanced six-phase system are shown in Fig. 110.



For 1st, 7th, 13th etc   For 3rd, 9th, 15th etc.   For 5th, 11th, 17th etc.

FIG. 110.

From inspection of the table it is obvious that for a balanced six-phase system, all harmonics which are present, except those of triple frequency and multiples of this frequency, differ by 60 degrees in time phase among themselves.

The harmonics of triple frequency and multiples of this frequency are either in phase or in phase opposition among them-

selves. those in adjacent phases always being in phase opposition. Omitting the even harmonics and the harmonics of triple frequency and multiples of this frequency, the phase order of the others, beginning with the first harmonic, *i.e.*, the fundamental, alternates from the direct order, *i.e.*, 1–2–3–4–5–6, to the reverse order, *i.e.*, 1–6–5–4–3–2. These phase relations among the fundamentals and harmonics of the different orders are what would be expected and what would necessarily follow from the phase relations existing among the fundamentals and harmonics of a balanced three-phase system.

For a balanced star-connected, six-phase system, the line voltage for the fundamental or for any harmonic, except the even harmonics and those of triple frequency and multiples of this frequency, is equal to the phase voltage in magnitude. Refer to Figs. 109 and 110.

$$\bar{V}_{12} = -\bar{V}_{01} + \bar{V}_{02}$$

$(-\bar{V}_{01} = \bar{V}_{10} = \bar{V}_{04})$ and $\bar{V}_{02}$ are two equal voltages which are 120 degrees apart in phase. Therefore,

$$\begin{aligned} V_{12} = V_{line} &= 2V_{phase} \cos 60^\circ \\ &= 2 \times \frac{1}{2} V_{phase} = V_{phase} \end{aligned} \tag{35}$$

For the harmonics of triple frequency or multiples of this frequency in a balanced three-phase system, $-\bar{V}_{01}$ and $\bar{V}_{02}$ are in phase. Therefore, for a balanced star-connected, six-phase system,

$$\begin{aligned} \bar{V}_{12} &= -\bar{V}_{01} + \bar{V}_{02} \\ V_{line} &= 2V_{phase} \end{aligned} \tag{36}$$

There can be no voltages of triple frequency or multiples of this frequency in the voltage between *alternate* line terminals of a balanced star-connected, six-phase system. Consider terminals 1 and 3. The voltage between these terminals for star connection is (Fig. 109, page 339)

$$\bar{V}_{13} = -\bar{V}_{01} + \bar{V}_{03}$$

By referring to the table on page 340, it is evident that the voltage $\bar{V}_{13}$ is zero for the harmonics of triple frequency and multiples of this frequency.

For mesh connection, the voltage between terminals 1 and 3 is (Fig. 109, page 339)

$$\bar{V}_{13} = -(\bar{V}_{01} + \bar{V}_{02}) \tag{37}$$

By referring to the table, it is evident that this is zero for harmonics of triple frequency or multiples of this frequency.

Although there can be no harmonics of triple frequency or multiples of this frequency in the voltage between alternate terminals, *i.e.*, between the three-phase terminals, of a balanced six-phase system, there may be harmonics of triple frequency or multiples of this frequency in the voltage between any diametrical terminals, such as 1 and 4, of a balanced six-phase system.

Relations exist among the line and phase *currents* of a balanced *mesh-connected,* six-phase system, which are similar to those holding between the line and phase *voltages* of a balanced *star-connected,* six-phase system.

When mesh connection is used for an armature, there can be no short-circuit current in the armature, as there may be in the case of three-phase delta connection, since the vector sum of all the component voltages acting around a balanced mesh-connected, six-phase circuit is zero. By referring to the table on page 340, it is evident that the vector sum of the six-phase voltages acting around a balanced mesh-connected, six-phase circuit is zero for the fundamental and for each odd harmonic. It would also be zero for each even harmonic, with the exception of the sixth harmonic and any multiple of that harmonic, if even harmonics were present.

The diametrical voltage of a six-phase, mesh-connected circuit between any two diametrical terminals, such as terminals 1 and 4, is

$$\bar{V}_{14} = \bar{V}_{10} + \bar{V}_{20} + \bar{V}_{30} \tag{38}$$

By referring to the table on page 340, it is evident that for harmonics of triple frequency or multiples of this frequency, this is equal to $\bar{V}_{03}$ or $\bar{V}_{01}$.

For star connection, the diametrical voltage between any diametrical terminals, such as 1 and 4, for balanced conditions is

$$\bar{V}_{14} = -\bar{V}_{01} + \bar{V}_{04} \tag{39}$$

By referring to the table on page 340, it is evident that, for the third harmonic or any multiple of the third harmonic, $V_{14} = 2\bar{V}_{04}$.

Therefore, for mesh connection, the third-harmonic, six-phase line voltage is equal to the third-harmonic phase voltage. For star connection, it is double the phase voltage. The same statements hold for any multiple of the third. Balanced conditions are assumed.

In general, for a balanced six-phase system which contains no even harmonics, no harmonics are suppressed in the voltage between adjacent terminals. Any odd harmonic voltage that exists in the phase voltage is present in the voltage between adjacent terminals. The harmonics of triple frequency and multiples of this frequency that exist in the phase voltage are suppressed between alternate terminals, *i.e.*, between the three-phase terminals. For mesh connection, the phase and line voltages are identical. For star connection, phase and line voltages are the same only when the phase voltages contain no harmonics of triple frequency or multiples of this frequency. For star connection, the fundamental line and phase voltages are equal. The line and phase voltages for any given harmonic, except the third or its multiples, are also equal. For the third harmonic or any harmonic whose frequency is an odd multiple of triple frequency, the line voltage for star connection is double the phase voltage. It is equal to the phase voltage for mesh connection.

An Example Illustrating the Relations among the Voltages of a Six-phase, Mesh-connected System Having a Badly Distorted Wave Form.—A number of years ago, a type of synchronous converter was built in such a way that its phase voltage could be badly distorted by changing the flux distribution in the air gap in order to alter the root-mean-square alternating voltage obtained with a given pole flux. In this way it was possible to change the ratio of the alternating voltage and the direct voltage. Under certain conditions, the phase voltage, *i.e.*, the voltage between adjacent armature taps, was found to be

$$e = 325 \sin 377t + 110 \sin (1131t + 90°) + 50 \sin (1885t + 50°)$$

If time is reckoned from the instant when the fundamental in the voltage of the phase between armature taps 1 and 2 is zero and increasing in a positive direction, what are:

(*a*) The expressions for the six instantaneous six-phase voltages?

(*b*) The expression for the instantaneous three-phase voltage between armature taps 1 and 3?

(*c*) The expression for the instantaneous diametrical voltage between armature taps 1 and 4?

(*d*) What are the root-mean-square values of the six-phase voltages, *i.e.*, the voltages between adjacent armature taps; the three-phase voltages, *i.e.*, the voltages between alternate armature taps; and the diametrical voltages, *i.e.*, the voltages between any two diametrical armature taps such as 1 and 4?

The fundamental voltage between armature taps 1 and 2 is assumed to lead the fundamental voltage between taps 2 and 3.

The armature of a synchronous converter is mesh-connected.

The expressions for the six instantaneous, six-phase voltages are (see Figs. 109 and 110, pages 339 and 340, respectively, and also the table on page 340)

$$\begin{aligned}
e_{12} = e_{10} &= 325 \sin 377t + 110 \sin (1131t + 90°) \\
&\quad + 50 \sin (1885t + 50°) \\
e_{23} = e_{20} &= 325 \sin (377t - 60°) + 110 \sin (1131t - 180° + 90°) \\
&\quad + 50 \sin (1885t - 300° + 50°) \\
e_{34} = e_{30} &= 325 \sin (377t - 120°) + 110 \sin (1131t - 0° + 90°) \\
&\quad + 50 \sin (1885t - 240° + 50°) \\
e_{45} = e_{40} &= 325 \sin (377t - 180°) + 110 \sin (1131t - 180° + 90°) \\
&\quad + 50 \sin (1885t - 180° + 50°) \\
e_{56} = e_{50} &= 325 \sin (377t - 240°) + 110 \sin (1131t - 0° + 90°) \\
&\quad + 50 \sin (1885t - 120° + 50°) \\
e_{61} = e_{60} &= 325 \sin (377t - 300°) + 110 \sin (1131t - 180° + 90°) \\
&\quad + 50 \sin (1885t - 60° + 50°)
\end{aligned}$$

The root-mean-square value of the six-phase voltage is

$$\begin{aligned}
V_{six\text{-}phase} &= \sqrt{\frac{(325)^2 + (110)^2 + (50)^2}{2}} \\
&= 245.2 \text{ volts}
\end{aligned}$$

The expression for the instantaneous voltage between armature taps 1 and 3 is (see Fig. 109, page 339)

$$\begin{aligned}
e_{13} &= e_{10} + e_{20} \\
&= \sqrt{3} \times 325 \sin (377t - 30°) + 0 \\
&\quad + \sqrt{3} \times 50 \sin (1885t + 50° + 30°) \\
&= 562.9 \sin (377t - 30°) + 86.6 \sin (1885t + 80°)
\end{aligned}$$

The root-mean-square value of the voltage between armature taps 1 and 3 is

$$\begin{aligned}
V_{13} = V_{three\ phase} &= \sqrt{\frac{(562.9)^2 + (86.6)^2}{2}} \\
&= 402.7 \text{ volts}
\end{aligned}$$

The ratio of the three-phase and six-phase voltages is

$$\frac{V_{three\text{-}phase}}{V_{six\text{-}phase}} = \frac{402.7}{245.2} = 1.642$$

When there are no even harmonics and no third harmonics, or multiples of the third, present in the six-phase voltage, the ratio of the three-phase and six-phase voltages is equal to 1.732, *i.e.*, the square root of three.

The voltage between any two diametrical armature taps is equal to the vector sum of three equal vectors which are displaced by equal angles. The easiest way to find the resultant of these three vectors is to add the first and third vectors and then add their resultant to the second vector. For the fundamental and all harmonics, except those of triple frequency or a multiple of this frequency, the first and third vectors are 120 degrees apart in time phase. The vector sum of these is equal both in phase and in magnitude to the second vector. The vector sum of the three vectors, therefore, is in phase with the second vector but has twice its magnitude. That this statement is true is evident from Fig. 110, page 340. For the harmonics of triple frequency or for harmonics of any odd multiple of triple frequency, the first and second vectors are opposite in phase. (See Fig. 110, page 340.) Their vector sum is zero. The resultant of the three vectors for the harmonics of triple frequency or for harmonics of any odd multiple of triple frequency is equal in direction and magnitude to the third vector.

It follows from the preceding statements that the expression for the instantaneous voltage between diametrical armature taps 1 and 4 is

$$
\begin{aligned}
e_{14} &= e_{10} + e_{20} + e_{30} \\
&= 2 \times 325 \sin (377t - 60^\circ) \\
&\qquad\qquad + 110 \sin (1131t + 90^\circ - 180^\circ) \\
&\qquad\qquad\qquad\qquad + 2 \times 50 \sin (1885t + 50^\circ + 60^\circ) \\
&= 650 \sin (377t - 60^\circ) + 110 \sin (1131t - 90^\circ) \\
&\qquad\qquad\qquad\qquad + 100 \sin (1885t + 110^\circ)
\end{aligned}
$$

The root-mean-square value of the voltage between diametrical armature taps 1 and 4 is

$$
\begin{aligned}
V_{14} = V_{diametrical} &= \sqrt{\frac{(650)^2 + (110)^2 + (100)^2}{2}} \\
&= 471.5 \text{ volts}
\end{aligned}
$$

The ratio of the diametrical and six-phase voltages is

$$
\frac{V_{diametrical}}{V_{six\text{-}phase}} = \frac{471.5}{245.2} = 1.923
$$

When there are no even harmonics and no harmonics of triple frequency or multiples of this frequency present in the voltages, the ratio of the diametrical and six-phase voltages is equal to 2.

It should be noted that the wave forms of the six-phase, three-phase and diametrical voltages are all different.

CHAPTER XII

POWER AND POWER FACTOR OF POLYPHASE CIRCUITS, RELATIVE AMOUNTS OF COPPER REQUIRED FOR POLYPHASE CIRCUITS, POWER MEASUREMENTS IN POLYPHASE CIRCUITS

Power and Power Factor of Balanced Polyphase Circuits.—Since a polyphase alternator has as many independent windings on its armature as it has phases, it is evident that the total output of such an alternator must be equal to the sum of the outputs of all its phases, no matter how they may be interconnected. In general, the power in any polyphase circuit whatsoever is equal to the sum of the powers developed in each phase.

$$\begin{aligned}\text{Total power} = P_0 &= P_1 + P_2 + P_3 + \text{etc.} \\ &= \Sigma P\end{aligned} \tag{1}$$

where the P's with subscripts 1, 2, 3 etc. are the powers developed in phases 1, 2, 3 etc., respectively.

For a three-phase circuit,

$$\begin{aligned}P_0 &= P_1 + P_2 + P_3 \\ &= V_1I_1 \times (p.f.)_1 + V_2I_2 \times (p.f.)_2 \\ &\qquad + V_3I_3 \times (p.f.)_3\end{aligned} \tag{2}$$

V_1, V_2 and V_3 are the phase voltages, I_1, I_2 and I_3 are the corresponding phase currents, and $(p.f.)_1$, $(p.f.)_2$ and $(p.f.)_3$ are the corresponding phase power factors. For sinusoidal waves, the power factor is equal to the cosine of the angle between the *phase* current and the *phase* voltage.

In a balanced polyphase circuit, all the phase currents are equal in magnitude and differ in phase by $\frac{360}{n}$ degrees, where n is the number of phases. All phase voltages are also equal in magnitude and differ in phase by $\frac{360}{n}$ degrees, except for two-phase.

It follows for a balanced polyphase circuit that the phase power factors must all be equal.

For a balanced circuit,

$$\text{Total power} = P_0 = nP_p = nV_pI_p \times (p.f.)_p \tag{3}$$

where n is the number of phases and V_p, I_p and $(p.f.)_p$ are the phase voltage, the phase current and the phase power factor, respectively.

For a balanced three-phase circuit,

$$P_0 = 3P_p = 3V_pI_p \times (p.f.)_p \tag{4}$$

$$(p.f.)_p = \frac{3P_p}{3I_pV_p} = \frac{P_p}{I_pV_p} \tag{5}$$

For a Δ-connected circuit, the line and phase voltages are the same. For a Y-connected circuit, the line and phase currents are the same. For a balanced circuit having sinusoidal waves of current and voltage, the line current is equal to the phase current multiplied by the square root of three for delta connection. The line voltage is equal to the phase voltage multiplied by the square root of three for wye connection.

Indicating line current and line voltage by I_L and V_L, respectively, equations (4) and (5) for sinusoidal waves and a balanced Δ-connected circuit become

$$\begin{aligned} \text{Total power} = P_0 &= 3V_p\frac{I_L}{\sqrt{3}}\,(p.f.)_p \\ &= \sqrt{3}V_LI_L \cos\theta_p \end{aligned} \tag{6}$$

$$(p.f.)_p = \cos\theta_p = \frac{P_0}{3\frac{I_L}{\sqrt{3}}V_L} = \frac{P_0}{\sqrt{3}I_LV_L} \tag{7}$$

where θ_p is the phase power-factor angle, *i.e.*, the phase angle between any *phase* current and the corresponding *phase* voltage.

For sinusoidal waves and a balanced Y-connected circuit, equations (4) and (5) become

$$\begin{aligned} \text{Total power} = P_0 &= 3\frac{V_L}{\sqrt{3}}\,I_p(p.f.)_p \\ &= \sqrt{3}V_LI_L \cos\theta_p \end{aligned} \tag{8}$$

$$(p.f.)_p = \cos\theta_p = \frac{P_0}{3I_L\frac{V_L}{\sqrt{3}}} = \frac{P_0}{\sqrt{3}I_LV_L} \tag{9}$$

From equations (6), (7), (8) and (9), it is evident that the expressions for the total power and the power factor of a balanced three-phase circuit, having sinusoidal waves of current and voltage, are the same whether the circuit is Δ- or Y-connected.

Equations (6), (7), (8) and (9) also hold for balanced conditions when the current and voltage waves are not sinusoidal, provided there are no harmonics present of triple frequency or any multiple of that frequency. This follows from the fact that when a balanced three-phase circuit contains no harmonics of triple frequency or multiples of this frequency, the ratio of line to phase voltage for wye connection and the ratio of line to phase current for delta connection are equal to the square root of three. (See table on page 318.) When equations (6), (7), (8) and (9) are applied to a balanced circuit having non-sinusoidal waves but no harmonics of triple frequency or any multiple of this frequency, V_L and I_L must be understood to be the equivalent sinusoidal voltage and current, respectively. The angle θ_p is the equivalent phase angle for the equivalent sinusoidal waves in each phase.

Since there can be no third harmonics in the line voltages or line currents of a balanced three-phase system, there can be no third harmonics in the equivalent wye voltages or equivalent delta currents of such a system. There are only a few cases where there are third harmonics in the actual wye voltages of a circuit or in the actual delta currents of a circuit, when balanced line voltages are impressed.

From equations (7) and (9), it is obvious that the power factor of a balanced three-phase circuit, which does not contain harmonics of triple frequency or multiples of triple frequency in either voltage or current in any of its phases, is given by

$$(p.f.)_p = \frac{P_0}{\sqrt{3}V_LI_L} \tag{10}$$

For sinusoidal waves of voltage and current, this is equal to the cosine of the phase angle between the *phase* current and

phase voltage. Power factor is *never* the cosine of the phase angle between the line current and the line voltage.

The power factor of a balanced four-phase circuit is given by the expression

$$(p.f.)_p = \frac{4V_pI_p \times (p.f.)_p}{4V_pI_p} = \frac{P_0}{4V_pI_p} \tag{11}$$

When no even harmonics are present, for star connection and balanced conditions, the line voltage is equal to the phase voltage multiplied by the square root of two, and the line and phase currents are equal. (See table on page 332.) This statement is independent of the wave form of the current or voltage. Therefore, for a balanced four-phase, star-connected circuit which has no even harmonics,

$$(p.f.)_p = \frac{P_0}{4\dfrac{V_L}{\sqrt{2}}I_L} = \frac{P_0}{2\sqrt{2}V_LI_L} \tag{12}$$

Under similar conditions, the same expression holds for a balanced mesh-connected, four-phase circuit. For a balanced mesh-connected, four-phase circuit which contains no even harmonics, the line current is equal to the phase current multiplied by the square root of two, and the line and phase voltages are the same. This statement is independent of the wave form of the current or voltage. Therefore, for a balanced four-phase, mesh-connected circuit which contains no even harmonics,

$$(p.f.)_p = \frac{P_0}{4V_L\dfrac{I_L}{\sqrt{2}}} = \frac{P_0}{2\sqrt{2}V_LI_L} \tag{13}$$

Equations (12) and (13) are true for any balanced four-phase circuit, whether star- or mesh-connected, and are independent of wave form, provided no even harmonics are present. Even harmonics are seldom present in power circuits. For sinusoidal waves of current and voltage, equations (12) and (13) reduce to $\cos \theta_p$, where θ_p is the phase angle between the *phase* current and the *phase* voltage.

Power Factor of an Unbalanced Polyphase Circuit.—The power factor of a balanced polyphase circuit is a perfectly definite thing which may be determined by simple measurements.

For a balanced polyphase circuit, the power factors of all phases are equal. The power factor of an unbalanced polyphase circuit may be defined in a number of ways, but the definition adopted must be workable, must have significance and must be useful. The power factor defined as the average power factor of the phases would be of no use, and in most cases it would be difficult to determine.

The power factor of a single-phase circuit having sinusoidal current and voltage waves may be written

$$p.f. = \cos\theta = \frac{P}{\sqrt{P^2 + Q^2}} = \frac{VI\cos\theta}{\sqrt{(VI\cos\theta)^2 + (VI\sin\theta)^2}} \tag{14}$$

where P and Q are the active and the reactive powers of the circuit. (See page 73.) In case the current and voltage waves are not sinusoidal, equation (14) has little physical significance.

The power factor of a balanced polyphase circuit having sinusoidal current and voltage waves may be similarly written

$$(p.f.)_p = \cos\theta_p = \frac{nP}{\sqrt{(nP)^2 + (nQ)^2}} \tag{15}$$

where n is the number of phases. The nP and the nQ are, respectively, the total active power and the total reactive power of the circuit. According to equation (15), the power factor of a balanced polyphase circuit is equal to the ratio of the total active power to the square root of the sum of the squares of the total active power and the total reactive power. This definition may be used to define the power factor of an unbalanced polyphase circuit. According to this definition, the power factor of an unbalanced polyphase circuit having sinusoidal current and voltage waves is

$$p.f. = \frac{\Sigma VI\cos\theta}{\sqrt{(\Sigma VI\cos\theta)^2 + (\Sigma VI\sin\theta)^2}} \tag{16}$$

In the case of a three-phase circuit,

$$\Sigma VI\cos\theta = V_1I_1\cos\theta_1 + V_2I_2\cos\theta_2 + V_3I_3\cos\theta_3 \tag{17}$$

$$\Sigma VI\sin\theta = V_1I_1\sin\theta_1 + V_2I_2\sin\theta_2 + V_3I_3\sin\theta_3 \tag{18}$$

where the subscripts 1, 2 and 3 are used with V, I and θ to indicate the phase voltages, the phase currents and the phase power-factor angles of the phases.

The expression $\sqrt{(\Sigma VI \cos \theta)^2 + (\Sigma VI \sin \theta)^2}$ is known as the vector volt-amperes of the circuit. It is equal to the magnitude of the resultant vector found by adding the volt-amperes of the phases as vectors referred to a common axis, using the power-factor angles of the phases as the phase angles for the vectors. For a balanced load, the vector volt-amperes are equal to the volt-amperes of the phases multiplied by the number of phases. This is the same as the algebraic sum of the volt-amperes of the separate phases.

Equation (16) is the second definition for power factor of a polyphase circuit recommended at the convention of the American Institute of Electrical Engineers, July, 1920,[1] by a joint committee of the American Institute of Electrical Engineers and the National Electric Light Association. It is the definition for the power factor given in the Revised Report on Standard Definitions and Symbols of the American Institute of Electrical Engineers, June, 1929.

Relative Amounts of Copper Required to Transmit a Given Amount of Power a Fixed Distance, with a Fixed Line Loss and Fixed Voltage between Conductors, over a Three-phase Transmission Line under Balanced Conditions and over a Single-phase Line.—Let P and V be, respectively, the power transmitted and the limiting voltage between conductors. Let I_1 and I_3 be the currents per conductor for the single-phase and three-phase transmission, respectively. Then,

$$P(\text{single-phase}) = VI_1 \cos \theta_p \tag{19}$$

$$P(\text{three-phase}) = \sqrt{3}VI_3 \cos \theta_p \tag{20}$$

where θ_p is the power-factor angle, *i.e.*, the angle between the *line* current and *line* voltage for single-phase transmission and the angle between the *line* current and the *equivalent wye* voltage for the three-phase transmission. Balanced three-phase transmission is assumed.

From equations (19) and (20),

$$\frac{I_3}{I_1} = \frac{1}{\sqrt{3}} \tag{21}$$

[1] Trans., A.I.E.E., vol. XXXIX, p. 1450.

Let r_1 and r_3 be the resistance per conductor of the single-phase and three-phase lines, respectively. Then, for equal copper losses for the single-phase and three-phase transmission,

$$2I_1^2r_1 = 3I_3^2r_3$$

$$\frac{r_1}{r_3} = \frac{3}{2} \times \frac{I_3^2}{I_1^2} = \frac{3}{2} \times \frac{1}{3} = \frac{1}{2} \qquad (22)$$

Since the amount of copper required for a given length of transmission line is directly proportional to the number of conductors employed and inversely proportional to the resistance per conductor,

$$\frac{\text{Copper for three-phase transmission}}{\text{Copper for single-phase transmission}} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4} \qquad (23)$$

From equation (23) it is evident that twenty-five per cent less copper is required to transmit a given amount of power a given distance, with a fixed transmission loss (copper loss) and a fixed voltage between conductors, over a three-phase line under balanced conditions than over a single-phase line, or thirty-three and a third per cent more copper is required for the single-phase transmission.

Relative Amounts of Copper Required to Transmit a Given Amount of Power a Fixed Distance, with a Fixed Line Loss and a Fixed Voltage between Conductors, over a Four-phase Transmission Line under Balanced Conditions and over a Single-phase Line.—The maximum voltage between conductors of a four-phase line exists between alternate conductors. If V represents this voltage, the voltage between adjacent conductors is

$$\frac{\sqrt{2}}{2} \times V = 0.707V$$

If the limiting voltage, V, is taken as the maximum voltage between any two of the conductors of the four-phase line, the four-phase system requires the same amount of copper as a single-phase line for the transmission of a given amount of power, under balanced conditions, a fixed distance with a fixed line loss. The four-phase system is equivalent to two single-phase lines, each transmitting half the total power at a voltage equal to the diametrical voltage of the system. Twice as many conductors are required as for single-phase transmission, but

each conductor need be only half as large, as it carries only half as much current as each conductor of the single-phase line.

If the voltage between adjacent conductors of the four-phase line is made equal to the voltage V between conductors of the single-phase line, the maximum voltage between any two conductors of the four-phase line is $\sqrt{2} \times V$ and occurs between alternate conductors.

If V is taken as the voltage between adjacent conductors of the four-phase line, and I_4 is used as the current per conductor,

$$P \text{ (four-phase)} = 4 \frac{V}{\sqrt{2}} I_4 \cos \theta_p$$

$$= 2\sqrt{2} V I_4 \cos \theta_p \tag{24}$$

$$P \text{ (single-phase)} = V I_1 \cos \theta_p \tag{25}$$

Since the power is the same in the two cases,

$$\frac{I_4}{I_1} = \frac{1}{2\sqrt{2}} \tag{26}$$

If r_4 is the resistance per conductor of the four-phase line, for equal losses for single-phase and four-phase transmission,

$$2 I_1^2 r_1 = 4\, I_4^2 r_4$$

$$\frac{r_1}{r_4} = 2 \times \frac{I_4^2}{I_1^2} = 2 \times \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{4} \tag{27}$$

$$\frac{\text{Copper for four-phase transmission}}{\text{Copper for single-phase transmission}} = \frac{4}{2} \times \frac{1}{4} = \frac{1}{2} \tag{28}$$

Relative Amounts of Copper Required to Transmit a Given Amount of Power a Fixed Distance, with a Fixed Line Loss and a Fixed Voltage to Neutral, when the Loads Are Balanced.—When the voltage to neutral is fixed, there is no difference in the amounts of copper required to transmit a given amount of power a fixed distance with a fixed line loss, whether the transmission is single-phase, three-phase or four-phase. Let V_n be the voltage to neutral. Then,

$$P \text{ (single-phase)} = 2 V_n I_1 \cos \theta_p \tag{29}$$

$$P \text{ (three-phase)} = 3 V_n I_3 \cos \theta_p \tag{30}$$

$$P \text{ (four-phase)} = 4 V_n I_4 \cos \theta_p \tag{31}$$

$$\frac{I_3}{I_1} = \frac{2}{3} \qquad \frac{I_4}{I_1} = \frac{2}{4}$$

$$2I_1^2r_1 = 3I_3^2r_3$$

$$\frac{r_1}{r_3} = \frac{3}{2} \times \frac{I_3^2}{I_1^2} = \frac{3}{2} \times \frac{4}{9} = \frac{2}{3}$$

$$2I_1^2r_1 = 4I_4^2r_4$$

$$\frac{r_1}{r_4} = \frac{4}{2} \times \frac{I_4^2}{I_1^2} = \frac{4}{2} \times \frac{4}{16} = \frac{1}{2}$$

$$\frac{\text{Copper for three-phase transmission}}{\text{Copper for single-phase transmission}} = \frac{3}{2} \times \frac{2}{3} = 1 \qquad (32)$$

$$\frac{\text{Copper for four-phase transmission}}{\text{Copper for single-phase transmission}} = \frac{4}{2} \times \frac{1}{2} = 1 \qquad (33)$$

Power Measurements in Three-phase Circuits.—The total power in any polyphase circuit is equal to the algebraic sum of

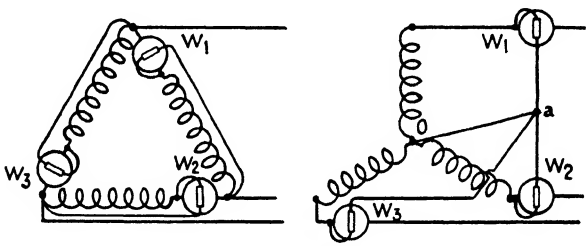


FIG. 111.

the powers in each phase. If a wattmeter is placed in each phase, the sum of the wattmeter readings is the true power in the circuit. Consider the three-phase, Δ-connected and Y-connected circuits shown in Fig. 111. The circles marked W in this figure represent the current coils of the wattmeters. The rectangles in the middle of the circles are the potential coils.

The current coil of each wattmeter carries the current in one phase. The voltage of this phase is impressed across the potential coil of the wattmeter. Obviously, each wattmeter, connected as shown in Fig. 111, measures the power in a single phase. The sum of the readings of the three wattmeters, for either the delta connection or the wye connection, must give the total power in the circuit.

Under ordinary conditions, it is impossible to break into a Δ-connected circuit and thus place a wattmeter in each phase as

shown in the left-hand half of Fig. 111. Neither is it always possible, in the case of a Y-connected circuit, to get at the neutral point which is required for the connections shown in the right-hand half of Fig. 111. If the connection shown in the right-hand diagram, between the common junction of the wattmeter potential coils and the neutral point of the circuit, is omitted, the sum of the readings of the three wattmeters—really the algebraic sum—gives the true power in the circuit for any degree of unbalancing or for any wave form, whether the circuit is Δ-connected or Y-connected, provided the load, if Y-connected, does not have its neutral point connected to the neutral of the source of power. If the neutrals are interconnected, but the neutral connection does not carry current, the conditions are the same as if it did not exist. Even with a balanced load, the neutral connection, if it exists, carries current if the phase voltages of the circuit contain harmonics of triple frequency or any multiple of triple frequency. When there are no harmonics of triple frequency or multiple of this frequency in the phase voltages of the source of power, there is still a pronounced third-harmonic current in the neutral connection under balanced conditions, when the load is inductive and contains iron, especially if the iron is worked at high saturation.

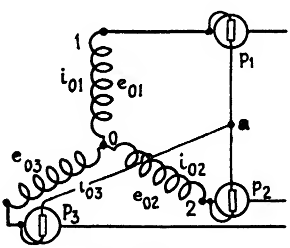


Fig. 112.

Proof of the Three-wattmeter Method for Measuring the Power in a Three-phase Circuit.—Let the instantaneous values of the phase currents, phase voltages and phase powers be denoted by i, e and p, respectively, with subscripts 1, 2 and 3 to indicate the different phases. Let the three wattmeters be connected as shown in Fig. 112.

The instantaneous power in any single-phase circuit is equal to the product of the instantaneous values of current and voltage. Therefore, the total instantaneous power p_0 in the three-phase circuit is

$$p_0 = i_{01}e_{01} + i_{02}e_{02} + i_{03}e_{03} \tag{34}$$

According to Kirchhoff's laws,

$$i_{01} + i_{02} + i_{03} = 0 \tag{35}$$

Let e_{0a} be the instantaneous potential difference between the points o and a, Fig. 112, *i.e.*, the difference in potential between the neutral point of the system and the common junction of the potential coils of the wattmeters. Then the following relation is obviously true:

$$e_{0a}(i_{01} + i_{02} + i_{03}) = 0$$
$$e_{0a}i_{01} + e_{0a}i_{02} + e_{0a}i_{03} = 0 \tag{36}$$

Subtracting equation (36) from equation (34) gives

$$i_{01}(e_{01} - e_{0a}) + i_{02}(e_{02} - e_{0a}) + i_{03}(e_{03} - e_{0a}) = p_0 \tag{37}$$

But

$$e_{01} - e_{0a} = e_{a1}$$
$$e_{02} - e_{0a} = e_{a2}$$
$$e_{03} - e_{0a} = e_{a3}$$

Therefore, the instantaneous power in the circuit is

$$p_0 = i_{01}e_{a1} + i_{02}e_{a2} + i_{03}e_{a3} \tag{38}$$
$$= p_1 + p_2 + p_3 \tag{39}$$

where p_1, p_2 and p_3 are the instantaneous powers corresponding to $i_{01}e_{a1}$, $i_{02}e_{a2}$ and $i_{03}e_{a3}$, respectively. The average power in the circuit is

$$P_0 = \frac{1}{T}\int_0^T p_1dt + \frac{1}{T}\int_0^T p_2dt + \frac{1}{T}\int_0^T p_3dt \tag{40}$$

which is the sum of the actual wattmeter readings.

This method of measuring power in a three-phase circuit is known as the *three-wattmeter method.* Since any Δ-connected circuit may always be replaced by an equivalent Y-connected circuit, it is evident that the three-wattmeter method of measuring power may be used to measure the total power in a Δ-connected circuit as well as to measure the total power in a Y-connected circuit.

The proof of the three-wattmeter method is based on the assumption that the sum of the instantaneous currents in the phases of the load is zero. If there is a neutral connection between the load and the source of power, and it carries current, the sum of the three instantaneous phase currents is no longer equal to zero. When the neutrals of the load and source of power

are connected and the neutral connection carries a current, i_{0n}, equation (35) becomes

$$i_{0n} + i_{01} + i_{02} + i_{03} = 0$$

The sum of the currents i_{01}, i_{02} and i_{03} is no longer zero, except when the neutral current is zero. The proof of the three-wattmeter method for measuring power holds only when there is no neutral connection between the source of power and the load, or when the neutral connection, if used, carries no current. If the neutral point a of the wattmeters is connected to the neutral point, o, of the load (Fig. 112, page 356), the three-wattmeter method of measuring power gives the true power under all conditions, since there is a wattmeter in each phase of the load.

The N-wattmeter Method for Measuring the Power in an N-phase Circuit.—The proof just given is not limited to three-phase circuits. It may be extended to apply to a circuit with any number of phases. The power in an n-phase circuit may be measured by n wattmeters, each with its current coil in one line and with its potential coil connected between the line in which its current coil is placed and a common junction point, to which one terminal of each wattmeter is connected. The n-wattmeter method does not hold when there is a neutral connection, which carries current, between the load and the source of power, except when the common junction of the potential coils of the n wattmeters is connected to the neutral of the load. The three-wattmeter method for measuring three-phase power or, in general, the n-wattmeter method for measuring n-phase power is seldom used. Three-phase power is usually measured by the two-wattmeter method. In general, the $(n - 1)$ wattmeter method is used for measuring the power in an n-phase circuit.

Two-wattmeter Method for Measuring Power in a Three-phase Circuit and the (N − 1)-Wattmeter Method for Measuring Power in an N-phase Circuit.—No assumption was made in the proof of the three-wattmeter method regarding the position of the common junction point a of the wattmeter potential coils, which may be any point whatsoever. It may be on one of the lines. In this case, one wattmeter reads zero and may be omitted. The algebraic sum of the readings of the two remaining wattmeters then gives the true power. If there are more than

three phases, the common junction point of the potential coils of the n wattmeters may still be on one of the lines, making the reading of one of the n wattmeters zero. The algebraic sum of the readings of the other $(n - 1)$ wattmeters is equal to the true power in the circuit.

In general, the power in an n-phase circuit may be measured by $(n - 1)$ wattmeters, each with its current coil in one line and its potential coil bridged between the line containing its current coil and the line which does not contain the current coil of a wattmeter. Applied to a three-phase circuit, the $(n - 1)$ wattmeter method becomes the two-wattmeter method. The connections for the two-wattmeter method for measuring power in a three-phase circuit are shown in Fig. 113.

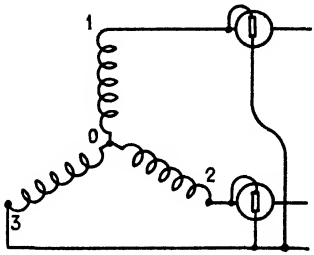


FIG. 113.

$$P_0 = \frac{1}{T}\int_0^T i_{01}e_{31}dt + \frac{1}{T}\int_0^T i_{02}e_{32}dt$$

The two-wattmeter method, or in general, the $(n - 1)$-wattmeter method, gives the true power in a circuit without regard to balance or wave form, provided the neutral of the load, if star-connected, does not have its neutral connected to the neutral of the source of power. If the neutral connection carries no current, the conditions are the same as if the neutral connection did not exist. Almost the only conditions likely to occur in practice, when a neutral connection for a three-phase circuit does not carry current, are when the load is exactly balanced and there are no harmonics present of triple frequency or any multiple of that frequency.

The connections of the wattmeters for the three-wattmeter method are symmetrical. The readings of the three wattmeters, for the three-wattmeter method of measuring the power in a three-phase circuit, therefore, are either all positive or all negative, except under unusual conditions, and add directly to give the true power in the circuit. The only case where the readings of the wattmeters are not either all positive or all negative is when the load is very badly unbalanced. Under all conditions, the algebraic sum of the wattmeter readings is the true power.

With the two-wattmeter method, the wattmeters are not connected symmetrically. By referring to Fig. 113, it is evident that wattmeter 1 has its potential coil connected from line 1 to line 3 or in a left-hand direction with respect to the sequence of phases as numbered. Wattmeter 2 has its potential coil connected from line 2 to line 3 or in a right-hand direction. The algebraic sum, not the numerical sum, of the readings of the two wattmeters always gives the true power. In order to read a wattmeter, it must be connected so that its pointer deflects up scale. It is necessary, therefore, to connect the two wattmeters in such a way that their pointers deflect up scale, and then to determine, from the connections used, whether the readings of the wattmeters are alike in sign or opposite. For balanced loads and sinusoidal waves of current and voltage, the readings are opposite in sign whenever the power factor of the circuit is less than five-tenths.

To determine whether the readings of the wattmeters used in the two-wattmeter method for measuring power in a three-phase circuit are of like or unlike sign, it is necessary merely to see whether the wattmeters are connected alike or differently, *i.e.*, to see whether the current is led to corresponding current terminals and whether corresponding ends of the potential coils are connected to the common line. If the wattmeters are connected alike and read up scale, their readings are both positive. If the wattmeters are of different make or type, they may both be placed in the same circuit to determine which terminals correspond. If the load is approximately balanced, the sign of the readings of the wattmeter may be determined easily by merely disconnecting from the common line (line 3 in Fig. 113) the potential coil of the wattmeter which has the smaller deflection and connecting it to the line not containing its current coil. If the reading of the wattmeter reverses when this is done, the signs of the wattmeter readings are opposite and the readings must be subtracted to give the true power. Changing the connections of the wattmeter with the smaller deflection, as indicated, connects the potential coils of both wattmeters alike with respect to the cyclic order of the three-phase circuit and serves the same purpose as placing both wattmeters in the same circuit.

The proofs for the n-wattmeter and the $(n-1)$-wattmeter methods for measuring power hold for any n-conductor circuit whatsoever. There is nothing in the proofs that limits them to polyphase circuits. For example, the power in an Edison three-wire, single-phase circuit, such as is commonly used to give 115 volts between the neutral and either outside conductor for lighting and 230 volts between the outside conductors for power, may be measured by the two-wattmeter method. The common way of metering the power in such a circuit is by a wattmeter with two elements which act on a common spindle. One current element is placed in each of the outside conductors. The potential elements are placed between the outside conductors and the neutral conductor.

The Total Reactive Power of a Three-phase Circuit, Whose Wave Forms Are Sinusoidal and Which Has No Neutral Connection That Carries Current, Measured by the Use of Two Reactive-power Meters Connected like the Wattmeters for Measuring Power by the Two-wattmeter Method for Measuring Power in a Three-phase Circuit.—Let the current and voltage waves of a three-phase circuit be sinusoidal. They may then be represented by vectors, each of which may be resolved into a real and a j component. Let small v's with and without primes represent, respectively, the j and real components of the voltages. Let small i's with and without primes represent, respectively, the corresponding components of the currents. Assume that there is no neutral current.

Refer to Fig. 113, page 359. Let

$$\bar{I}_{01} = i_1 + ji_1'$$
$$\bar{I}_{02} = i_2 + ji_2'$$
$$\bar{I}_{03} = i_3 + ji_3'$$

Also, let

$$\bar{V}_{01} = v_1 + jv_1'$$
$$\bar{V}_{02} = v_2 + jv_2'$$
$$\bar{V}_{03} = v_3 + jv_3'$$

Then, referring to page 72, the total reactive power is

$$P_r = (v_1'i_1 - v_1i_1') + (v_2'i_2 - v_2i_2') + (v_3'i_3 - v_3i_3') \qquad (41)$$

Since there is assumed to be no neutral current,

$$\bar{I}_{01} + \bar{I}_{02} + \bar{I}_{03} = 0$$

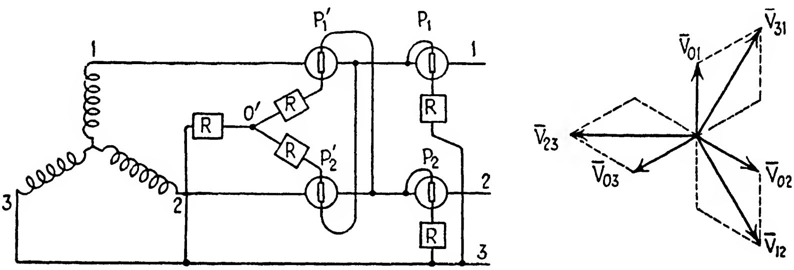
The following relations must also be true:

$$i_1 + i_2 + i_3 = 0 \tag{42}$$
$$i_1' + i_2' + i_3' = 0 \tag{43}$$

Substituting the values of i_3 and i_3' from equations (42) and (43) in equation (41) gives

$$\begin{aligned} P_r &= v_1'i_1 - v_1i_1' + v_2'i_2 - v_2i_2' - v_3'(i_1 + i_2) + v_3(i_1' + i_2') \\ &= i_1(v_1' - v_3') + i_1'(-v_1 + v_3) + i_2(v_2' - v_3') + i_2'(-v_2 + v_3) \\ &= (i_1v_{31}' - i_1'v_{31}) + (i_2v_{32}' - i_2'v_{32}) \qquad (44) \\ &= P_{1r} + P_{2r} \end{aligned}$$

where P_{1r} and P_{2r} are, respectively, the readings of reactive-power meters connected as shown in Fig. 113, page 359.



(a) (b)

FIG. 114.

Connections for Measuring the Reactive Power of a Three-phase Circuit.—Ordinary wattmeters may be used to measure the reactive power in a circuit, provided they can be connected in such a way that the potentials impressed on their potential coils are in quadrature with the potentials which would be applied if the wattmeters were connected for measuring power in the ordinary way. If the voltages of a three-phase circuit are balanced, two ordinary wattmeters may be connected so as to measure the total reactive power in the circuit.

Refer to Fig. 114*a*. P_1 and P_2 are two wattmeters which are connected according to the two-wattmeter method of measuring power. The squares marked R are the non-inductive resistances

which are always connected in series with the potential coils of wattmeters. Wattmeter P_1 gets the current $\bar{I}_{01}$ and the potential $\bar{V}_{31}$. Wattmeter P_2 gets the current $\bar{I}_{02}$ and the potential $\bar{V}_{32}$. If the potential coils can be reconnected in such a way that the potentials impressed on them are in quadrature with the potentials impressed when connected as shown, the algebraic sum of their readings is the total reactive power of the circuit. The wattmeters are shown reconnected at P_1' and P_2'. The two resistances marked R in series with the potential coils and a third resistance also marked R are in wye across the circuit. These resistances are assumed to be equal. Also assume that the voltages are balanced. Under these conditions, the point marked $0'$ is the true neutral for the voltages.

Refer to Fig. 114*a* and the vector diagram of the voltages which is shown in Fig. 114*b*. Wattmeter P_1' gets the same current as P_1, but the potential impressed on its potential coil is $\bar{V}_{0'2} = \bar{V}_{02}$, which lags $\bar{V}_{31}$ by 90 degrees. Wattmeter P_2' gets the same current as P_2, but the potential impressed on its potential coil is $\bar{V}_{1'0} = \bar{V}_{01}$, which leads $\bar{V}_{32}$ by 90 degrees. It should be noted that the connections of the potential coil P_2' should be reversed (not shown in the figure) with respect to the potential coil of P_1', in order to get both potentials lagging the potentials which would be applied when measuring power. One of the current coils might equally well be reversed, instead of one of the potential coils. If the magnitude of the potentials were the same as impressed on wattmeters P_1 and P_2, the algebraic sum of the readings of wattmeters P_1' and P_2' would be the total reactive power of the circuit.

Since the potentials on P_1' and P_2' are wye potentials instead of line potentials and are, therefore, $\frac{1}{\sqrt{3}}$ as great as for P_1 and P_2, the readings of P_1' and P_2' must be multiplied by the square root of three in order to get the total reactive power. By the proper graduation of the scales of the two meters, they may be made to read directly.

The wattmeter connections shown in Fig. 114*a* still give the true reactive power when the currents are not balanced, provided the voltages are balanced. The connections also give the reactive power when defined as $\Sigma VI \sin \theta$, provided there are no harmonics

of triple frequency or multiples of that frequency and also provided the line voltages are balanced.

Problem Illustrating the Use of the Two-wattmeter Method for Measuring Power in a Balanced Three-phase Circuit.—Two wattmeters are connected to measure the power taken by a certain balanced three-phase inductive load. Wattmeter 1 has its current coil in line 1 and its potential coil between lines 1 and 3. Wattmeter 2 has its current coil in line 2 and its potential coil between lines 2 and 3. The readings of the wattmeters are, respectively, 40.96 and 4.36 kilowatts. When that end of the potential coil of wattmeter 2 which is connected to line 3 is connected to line 1, the reading of the wattmeter 2 reverses and the pointer goes up against the stop. If the connection of the current coil of this wattmeter is now reversed, so that the pointer deflects up scale, the wattmeter reads 40.96 kilowatts.

(*a*) What is the power taken by the load?

(*b*) If the line currents are each 100 amperes and the voltages between lines are each 500 volts, what is the power factor of the load?

(*c*) By what angle does the equivalent sine current in each phase lag the equivalent sine voltage impressed across each phase?

(*d*) If the load is Y-connected without a neutral, what is the current in each phase and what is the voltage across the terminals of each phase?

(*e*) If the load had been Δ-connected, what would have been the current in each phase and what would have been the voltage across the terminals of each phase?

Since there is no neutral connection, when the load is connected in wye there can be no harmonic of triple frequency or multiple of this frequency in the current. There can be no harmonic of triple frequency or multiple of this frequency in the line voltage, since the load is balanced. There would be a third harmonic in the phase voltage, *i.e.*, the wye voltage, if the load involved iron. It is assumed that no iron is present. In this case, there can be no harmonic of triple frequency or multiple of this frequency in any part of the circuit. Under this condition, the ratio of the line and phase (wye) voltages is equal to the square root of three. If no iron is present, there is no

circulatory current of triple frequency or any multiple of this frequency in the closed delta, when delta connection is assumed. Under this condition, the ratio of line and phase (delta) currents is equal to the square root of three.

(a)

Since wattmeter 2 reverses when the connections of its potential coil are changed, its reading must have been negative as originally connected. Therefore,

$$\text{Total power} = P_0 = 40.96 - 4.36 = 36.60 \text{ kilowatts}$$

(b)

$$\text{p.f.} \quad \frac{P_0}{\sqrt{3}\, I_{line}\, V_{line}} = \frac{36{,}600}{\sqrt{3} \times 100 \times 500} = 0.4226$$

(c)

Angle of lag of the equivalent sine-wave current behind the equivalent sine-wave voltage for either wye or delta connection is

$$\theta_p = \cos^{-1} 0.4226 = 65 \text{ degrees}$$

(d)

$$\text{Wye connection} \begin{cases} I_{phase} = 100 \text{ amperes} \\ V_{phase} = \dfrac{500}{\sqrt{3}} = 288.7 \text{ volts} \end{cases}$$

(e)

$$\text{Delta connection} \begin{cases} I_{phase} = \dfrac{100}{\sqrt{3}} = 57.7 \text{ amperes} \\ V_{phase} = 500 \text{ volts} \end{cases}$$

Another Example of the Use of the Two-wattmeter Method.—It is frequently convenient, when solving problems in three-phase circuits, to apply the principle of the two-wattmeter method to determine the total power taken by a circuit. When the solution of a problem gives the line currents and line voltages expressed in their complex form, the easiest way to determine the total power consumed is to assume that wattmeters have been inserted in the circuit to measure the power by the two-wattmeter method.

The solution of the problem illustrating Kirchhoff's laws, which was given on page 305, gives for the three line currents

$$\bar{I}_{ao} = 20.10 - j10.44$$
$$\bar{I}_{bo} = -10.62 + j1.83$$
$$\bar{I}_{co} = -9.48 + j8.60$$

The line voltages are

$$V_{ab} = 230 + j0$$
$$V_{bc} = -115 - j199.2$$
$$V_{ca} = -115 + 199.2$$

Assume that one wattmeter is placed in line a (see Fig. 95, page 303), with its potential coil connected between lines a and b, and that the other wattmeter is placed in line c, with its potential coil connected between lines c and b. The algebraic sum of the readings of the wattmeters, if actually connected as indicated, is the total power in the circuit.

Wattmeter in line a carries a current $\bar{I}_{ao} = 20.10 - j10.44$ and has a voltage $\bar{V}_{ab} = 230 + j0$ impressed across its potential coil. This wattmeter reads

$$\begin{aligned} P_a &= (20.10)(230) + (-10.44)(0) \\ &= 4623 + 0 = 4623 \text{ watts} \end{aligned}$$

The other wattmeter carries a current $\bar{I}_{co} = -9.48 + j8.60$ and has a voltage $V_{cb} = -V_{bc} = 115 + j199.2$ impressed across its potential coil. This wattmeter reads

$$\begin{aligned} P_c &= (-9.48)(115) + (8.60)(199.2) \\ &= -1090 + 1713 = 623 \text{ watts} \\ \text{Total power } P_0 &= P_a + P_c = 4623 + 623 \\ &= 5246 \text{ watts} \end{aligned}$$

Relative Readings on Balanced Loads of Wattmeters Connected for the Two-wattmeter Method of Measuring Power in a Three-phase Circuit When the Current and Voltage Waves Are Sinusoidal.—Figure 115 shows the vector diagram of a balanced three-phase, inductive load having a power factor of $\cos \theta_p$. For an inductive load having a power factor of $\cos \theta_p$, the phase currents lag the phase voltages by an angle θ_p. I_{01}

lags $\bar{V}_{01}$ by θ_p degrees, $\bar{I}_{02}$ lags $\bar{V}_{02}$ by θ_p degrees and $\bar{I}_{03}$ lags $\bar{V}_{03}$ by θ_p degrees.

Let the wattmeters be connected as shown in Fig. 113, page 359. The wattmeter in line 1 carries the current $\bar{I}_{01}$ and has the voltage $\bar{V}_{31}$ impressed across the terminals of its potential coil. The voltage and current are considered in the same direction in phase 0-1. The wattmeter in line 2 carries the current $\bar{I}_{02}$ and has the voltage $\bar{V}_{32}$ impressed across the terminals of its potential coil. The current and voltage are considered in the same

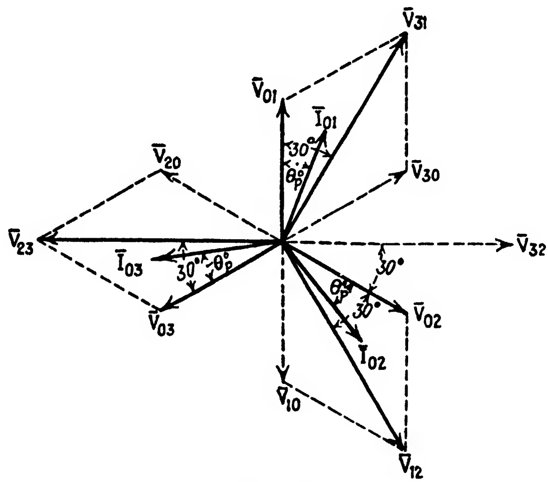


FIG. 115.

direction in phase 0-2. $\bar{I}_{01}$ lags $\bar{V}_{31}$ by $(\theta_p - 30)$ degrees. $\bar{I}_{02}$ lags $\bar{V}_{32}$ by $(\theta_p + 30)$ degrees. The readings of the two wattmeters are, therefore,

$$P_1(\text{wattmeter in line 1}) = I_{01}V_{31} \cos (\theta_p{}^\circ - 30^\circ)$$
$$= I_L V_L \cos (\theta_p{}^\circ - 30^\circ) \qquad (45)$$
$$P_2(\text{wattmeter in line 2}) = I_{02}V_{32} \cos (\theta_p{}^\circ + 30^\circ)$$
$$= I_L V_L \cos (\theta_p{}^\circ + 30^\circ) \qquad (46)$$

where the subscript L indicates line values.

When $\theta_p = 0$, *i.e.*, when the power factor of the circuit is unity, both wattmeters read alike, *viz.*, $I_L V_L \cos 30°$.

When $\theta_p = 60$ degrees, *i.e.*, when the power factor is 0.5, the current and voltage for the wattmeter in line 1 are out of

phase by 30 degrees, the current lagging the voltage, and the current and voltage for the wattmeter in line 2 are out of phase by 90 degrees, the current also lagging. The reading of the wattmeter in line 2 is therefore zero. The entire power developed in the circuit is indicated by the wattmeter in line 1. For angles of lag greater than 60 degrees, *i.e.*, for power factors less than 0.5, the current and voltage for the wattmeter in line 2 are out of phase by more than 90 degrees. Under this condition, the reading of the wattmeter in line 2 reverses, *i.e.*, it becomes negative and must be subtracted from the reading of the wattmeter in line 1 to get the true power. The true power is always equal to the algebraic sum of the readings of the two wattmeters, but the sign of the reading of one of the wattmeters reverses and becomes negative when the power factor of the circuit becomes less than 0.5.

When the power factor is less than 0.5, it is necessary to reverse the connections of the current coil of one of the wattmeters in order to make it read up scale. All readings taken after the reversal of the current coil must be considered negative.

The ratio of the readings of the two wattmeters, connected for the two-wattmeter method, is the same for equal power factors with leading and lagging currents, but the actual readings are interchanged. For example, the wattmeter in line 2 reads zero for a power factor of cos $(+60°)$, *i.e.*, a lag of the phase current behind the phase voltage of 60 degrees. For an angle of lead of 60 degrees, *i.e.*, a power factor of cos $(-60°)$, the wattmeter in line 1 reads zero.[1] Both wattmeters read alike for unity power factor. They read alike in magnitude but opposite in sign for zero power factor. It must be remembered that the above statements are true only when the load is balanced, and the current and voltage waves are sinusoidal.

Clockwise phase order is assumed in the preceding discussion. Changing the phase order interchanges the readings of the wattmeters.

The ratios of the readings of two wattmeters, connected to measure the power taken by a balanced three-phase load with sinusoidal current and voltage waves, are plotted against power factors in Fig. 116, page 369.

[1] Angles of lag are taken positive in equations 45 and 46.

Determination of the Power Factor of a Balanced Three-phase Circuit, When the Current and Voltage Waves Are Sinusoidal, from the Readings of Two Wattmeters Connected to Measure the Total Power by the Two-wattmeter Method.—The power factor of a balanced three-phase circuit may be determined by measuring the total power, the line current and the line voltage, and then applying equation (10), page 349. If a circuit is balanced and its current and voltage waves are sinusoidal,

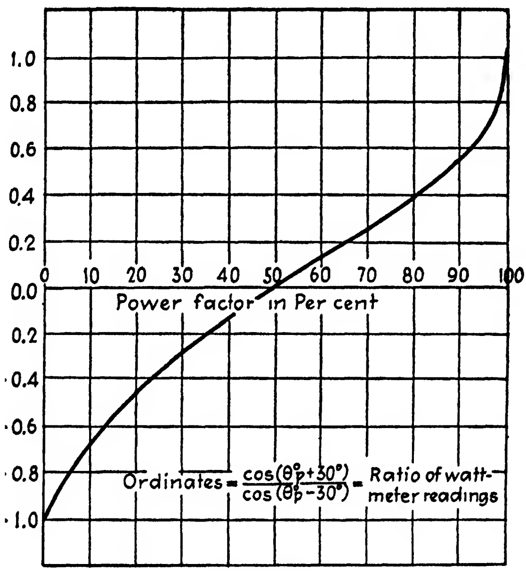


FIG. 116.

the power factor may be found from the readings of two wattmeters which are connected to measure the total power by the two-wattmeter method.

Refer to Fig. 115, page 367. From equations (45) and (46), page 367, the readings of the two wattmeters are

$$P_1 \text{ (wattmeter in line 1)} = I_L V_L \cos (\theta_p{}^\circ - 30^\circ)$$
$$P_2 \text{ (wattmeter in line 2)} = I_L V_L \cos (\theta_p{}^\circ + 30^\circ)$$

Expanding the cosine terms,

$$P_1 = I_L V_L\left(\frac{\sqrt{3}}{2} \cos \theta_p{}^\circ + \frac{1}{2} \sin \theta_p{}^\circ\right)$$

$$P_2 = I_L V_L\left(\frac{\sqrt{3}}{2}\cos\theta_p{}^\circ - \frac{1}{2}\sin\theta_p{}^\circ\right)$$

from which

$$\frac{P_1 - P_2}{P_1 + P_2} = \frac{\sin\theta_p{}^\circ}{\sqrt{3}\cos\theta_p{}^\circ}$$

$$\tan\theta_p{}^\circ = \sqrt{3}\,\frac{P_1 - P_2}{P_1 + P_2} \tag{47}$$

Since the tangent of the angle of lag between the phase current and phase voltage of a circuit is always equal to the ratio of the reactive phase power to the true phase power, it is obvious from equation (47) that $\sqrt{3}(P_1 - P_2)$ is reactive power. For balanced conditions, the square root of three times the difference of the readings of wattmeters connected to measure the power of a three-phase circuit by the two-wattmeter method is the reactive power of the circuit.

An Example Involving the Use of the Two-wattmeter Method for Measuring the Power in a Balanced Three-phase, Y-connected Circuit Containing Third Harmonics.—Each branch of a Y-connected load consists of a condenser, an air-core inductance and a non-inductive resistance, in series. When this circuit is connected to a source of power with balanced voltages which contain no harmonics of higher order than the third, each of two wattmeters, connected to measure power by the two-wattmeter method, reads 15 kilowatts. Each voltage between line terminals of the load is 230 volts. When the neutral of the Y-connected load is connected to the neutral of the source of power, the wattmeter readings and the voltages between the line terminals of the load remain unchanged, but there is a current in the neutral connection of 60 amperes. Under this condition, each of the voltages between the line terminals of the load and the neutral point is 150 volts. What are the resistance, inductance and capacitance of each branch of the load? The fundamental frequency of the impressed voltage is 60 cycles.

Let the subscripts 1 and 3 attached to the letters P, I and V indicate, respectively, fundamental and third-harmonic power, current and voltage per phase.

Since the line voltages of a balanced three-phase circuit cannot contain third harmonics, the 230 volts impressed across the

terminals of the load must contain the fundamental only. Therefore, the fundamental voltage to neutral must be

$$V_1 = \frac{230}{\sqrt{3}} = 132.8 \text{ volts}$$

Since the wattmeters have their potential coils connected across the line terminals of the load, the voltages impressed across the potential coils must contain the fundamentals only. The wattmeters can record only fundamental power, even if their current coils carry third-harmonic current, since their potential coils have only fundamental voltage impressed across them. There can be no power developed by a harmonic in the current if the corresponding harmonic in the voltage is absent.

Since the two wattmeters read alike, the power factor of the circuit for the fundamental alone must be unity. The circuit must, therefore, be in resonance for the fundamental.

$$\begin{aligned} P_1 &= V_1^2 g_1 \\ &= V_1^2 \frac{r_1}{r_1^2 + x_1^2} \end{aligned}$$

where g_1, r_1 and x_1 are the conductance, resistance and reactance per phase for the fundamental. For resonance, x_1 is zero and

$$P_1 = \frac{V_1^2}{r_1}$$

$$r_1 = \frac{V_1^2}{P_1} = \frac{(132.8)^2}{\frac{15{,}000 + 15{,}000}{3}}$$

$$= 1.76 \text{ ohms}$$

The third-harmonic voltage to neutral is

$$\begin{aligned} V_3 &= \sqrt{(150)^2 - (132.8)^2} \\ &= 69.8 \text{ volts} \end{aligned}$$

Third-harmonic current per phase is equal to one-third of the third-harmonic current in the neutral.

$$I_3 = \frac{I_{3n}}{3} = \frac{60}{3} = 20 \text{ amperes}$$

The third-harmonic impedance per phase is

$$z_3 = \frac{69.8}{20} = 3.49 \text{ ohms}$$

$$x_3 = \sqrt{(3.49)^2 - (1.76)^2}$$

$$= 3.01 \text{ ohms}$$

Let x_{L1} and x_{C1} represent the inductive and capacitive reactances for the fundamental. Let x_{L3} and x_{C3} represent the same quantities for the third harmonic. Then,

$$x_{L1} + x_{C1} = 0$$

$$x_{L3} + x_{C3} = 3x_{L1} + \frac{1}{3}x_{C1} = 3.01$$

$$x_{L1}(9 - 1) = 9.03$$

$$x_{L1} = 1.13 \text{ ohms}$$

$$x_{C1} = -1.13 \text{ ohms}$$

$$L = \frac{x_{L1}}{2\pi f_1} = \frac{1.13}{377} = 0.00300 \text{ henry}$$

$$C = \frac{-10^6}{2\pi f_1 x_{C1}} = \frac{-10^6}{377 \times (-1.13)} = 2347 \text{ microfarads}$$

Measurement of the Reactive Power of a Balanced Three-phase Circuit, Whose Current and Voltage Waves Are Sinusoidal, from the Readings of Two Wattmeters with the Connections of Their Potential Coils Interchanged.—If the potential coils of two wattmeters, which are arranged to measure the power of a balanced three-phase circuit whose current and voltage waves are sinusoidal, are interchanged, the reading of either wattmeter multiplied by the square root of three is equal to the reactive power in the circuit. For example, if the potential coil of the wattmeter with its current coil in line 1, Fig. 113, page 359, is connected between lines 2 and 3 instead of between lines 1 and 3, and the potential coil of the wattmeter with its current coil in line 2 is connected between lines 1 and 3 instead of between lines 2 and 3, the reading of each wattmeter is

$$I_L V_L \sin \theta_p$$

If the potential coils of the two wattmeters are reconnected as stated above, it is evident, by referring to Fig. 115, page 367, that the wattmeters read

$$
\begin{aligned}
P_1 &= I_{01}V_{32}\cos(90° - \theta_p°) \\
&= I_L V_L \sin \theta_p° \\
P_2 &= I_{02}V_{31}\cos(90° + \theta_p°) \\
&= I_L V_L \sin(-\theta_p°) \\
\text{Reactive power} &= 3I_{phase}V_{phase}\sin\theta_{phase} \\
&= 3I_p V_p \sin\theta_p \\
&= \sqrt{3}I_L V_L \sin\theta_p \\
&= \sqrt{3}P_1 = \sqrt{3}P_2
\end{aligned}
$$

CHAPTER XIII

SYMMETRICAL PHASE COMPONENTS APPLIED TO UNBALANCED THREE-PHASE CIRCUITS

Unbalanced Circuits.—The problems involved in the operation of balanced three-phase circuits or, in general, of balanced polyphase circuits are comparatively simple and easy of solution. Except when there is interaction between phases, such problems may be treated like any single-phase problem by considering merely one phase. Although in practice most polyphase circuits are nearly balanced, many cases of bad unbalancing exist. Moreover, certain types of polyphase apparatus are particularly sensitive to unbalancing. Problems involving unbalanced conditions and faults on transmission lines are of frequent occurrence in practice. These and many other problems make any method which simplifies their solution of great importance. Such a method, known as the *method of symmetrical phase components*, depends upon the fact that any system of unbalanced sinusoidal, polyphase currents or voltages may be resolved into systems of symmetrical components.[1]

As most polyphase power circuits and apparatus are three-phase, only the unbalanced three-phase system will be considered in detail. Any unbalanced system of three-phase vectors may be resolved into three systems of components. The components of the first of these systems are identical. They are known as the *zero-phase* or the *zero-sequence* components. They are sometimes called the *residuals*. The components of the second system together give rise to a balanced three-phase system whose

[1] R. E. Gilman and C. LeG. Fortescue, Trans., A.I.E.E., vol. XXXV, p. 1329, 1916.

C. L. Fortescue, Trans., A.I.E.E., vol. XXXVII, p. 1027, 1918.

W. V. Lyon, Unbalanced Three-phase Circuits, Electrical World, June 5, 1920.

Symmetrical Components, C. F. Wagner and R. D. Evans.

phase order is the *same* as the phase order of the original vectors. These are known as the *positive-phase* or *positive-sequence* components. The components of the third system together give rise to a balanced three-phase system, but the phase order of this system is *opposite* to that of the original vectors. These are known as the *negative-phase* or *negative-sequence* components. In other words, any unbalanced three-phase system of sinusoidal voltages may be replaced by two balanced systems of sinusoidal three-phase voltages, having opposite phase orders, and three sinusoidal zero-phase voltages. Any unbalanced system of three-phase currents, whose wave forms are sinusoidal, may be similarly replaced.

Positive-phase, Negative-phase and Zero-phase Components of Three-phase Vectors.—Let $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$ be the three voltages of an unbalanced three-phase system whose voltages are sinusoidal. Each of these voltages may be resolved into any number of sinusoidal components of the same frequency, but since there are only three known quantities, *viz.*, $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$, there can be only three independent simultaneous equations to determine the components. Let each voltage be resolved into three components. Then

$$\bar{V}_1 = \bar{x} + \bar{y} + \bar{z} \tag{1}$$

$$V_2 = \bar{a}\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{z} \tag{2}$$

$$\bar{V}_3 = \bar{d}\bar{x} + \bar{e}\bar{y} + \bar{f}\bar{z} \tag{3}$$

where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the three components of $\bar{V}_1$, and $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ are complex coefficients. These coefficients must be so related that they may be determined from equations (1), (2) and (3), which are the only possible simultaneous equations.

Since the operation of motors and generators, and three-phase systems in general, is well understood under balanced conditions and can be handled easily, two of the coefficients, $\bar{b}$ and $\bar{e}$, are so chosen that the components $\bar{y}$, $\bar{b}\bar{y}$ and $\bar{e}\bar{y}$ form a balanced three-phase system of voltages, whose phase order is the same as that of the original three voltages. The coefficients $\bar{c}$ and $\bar{f}$ are so fixed that the components $\bar{z}$, $\bar{c}\bar{z}$ and $\bar{f}\bar{z}$ also form a balanced three-phase system of voltages, but the phase order of this system of voltages is opposite to that of the original

vectors. The remaining coefficients, *viz.*, $\bar{a}$ and $\bar{d}$, are fixed so that the components $\bar{x}$, $\bar{a}\bar{x}$ and $\bar{d}\bar{x}$ are identical. Each of the coefficients $\bar{a}$ and $\bar{d}$ is therefore unity. Then,

$$\bar{V}_1 + \bar{V}_2 + \bar{V}_3 = 3\bar{x} + (1 + \bar{b} + \bar{e})\bar{y} + (1 + \bar{c} + \bar{f})\bar{z} \quad (4)$$

and

$$\bar{y} + \bar{b}\bar{y} + \bar{e}\bar{y} = 0 \quad (5)$$

$$\bar{z} + \bar{c}\bar{z} + \bar{f}\bar{z} = 0 \quad (6)$$

These conditions, together with the known values of $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$, are sufficient to fix the components definitely. Other systems might be chosen, but this would complicate the conditions produced by unbalancing rather than simplify them. The particular advantage of the systems chosen is that two are balanced systems, for which the conditions are well understood.

When the conditions are such that the vector sum of the three original vectors is zero, *i.e.*, when

$$\bar{V}_1 + \bar{V}_2 + \bar{V}_3 = 0$$

$3\bar{x}$ must also be zero, since $\bar{y} + \bar{b}\bar{y} + \bar{e}\bar{y}$ and $\bar{z} + \bar{c}\bar{z} + \bar{f}\bar{z}$ are each always equal to zero.

The vector sum of the three line voltages of any three-phase load is always zero. Hence there cannot be any zero-phase components in the line voltages of a three-phase load. There cannot be any zero-phase components in the line currents of either a balanced or unbalanced Δ-connected load or in the line or phase currents of a balanced or unbalanced Y-connected load without a neutral connection, since the vector sum of the currents in each case must be zero. There probably are zero-phase components in the phase voltages of an unbalanced Y-connected load either with or without a neutral connection, since the vector sum of these voltages, in general, is not zero. It is possible, however, to have an unbalanced Y-connected load in which the zero-phase components are zero. Since, in general, the vector sum of the phase currents in an unbalanced Δ-connected load is not zero, these currents usually contain zero-phase components. In this case, also, it is possible to have a load in which the zero-phase components are zero. There are zero-phase components in the line and phase currents of an unbalanced

Y-connected load with a neutral connection which carries current, since the vector sum of the phase currents for wye connection, which in this case are the line currents, is not zero.

Since $(1 + \bar{b} + \bar{e})\bar{y}$ and $(1 + \bar{c} + \bar{f})\bar{z}$ are each equal to zero, it follows that

$$\bar{V}_1 + \bar{V}_2 + \bar{V}_3 = 3\bar{x}$$

$$\bar{x} = \frac{\bar{V}_1 + \bar{V}_2 + \bar{V}_3}{3} \tag{7}$$

That is, the zero-phase component in each phase is equal to one-third of the vector sum of the three original vectors.

$$\bar{V}_1 - \bar{x} = \bar{y} + \bar{z} \tag{8}$$

$$\bar{V}_2 - \bar{x} = \bar{b}\bar{y} + \bar{c}\bar{z} \tag{9}$$

$$\bar{V}_3 - \bar{x} = \bar{e}\bar{y} + \bar{f}\bar{z} \tag{10}$$

The vector sum of the positive-phase and the negative-phase components for each phase may be found by subtracting the zero-phase component vectorially from each of the original voltage vectors.

Instead of using $\bar{x}$, $\bar{y}$ and $\bar{z}$ as the components of $\bar{V}_1$, and the coefficients $\bar{b}$, $\bar{c}$, $\bar{e}$ and $\bar{f}$, it will be simpler and more convenient in what follows to use the symbols 0, + and − in the position of exponents with the letter V to indicate the zero-phase, the positive-phase and the negative-phase components, respectively. Using this notation,

$$\bar{V}_1 = \bar{V}^0 + \bar{V}_1^+ + \bar{V}_1^- \tag{11}$$

$$\bar{V}_2 = \bar{V}^0 + \bar{V}_2^+ + \bar{V}_2^- \tag{12}$$

$$\bar{V}_3 = \bar{V}^0 + \bar{V}_3^+ + \bar{V}_3^- \tag{13}$$

$$\bar{V}_1 + \bar{V}_2 + \bar{V}_3 = 3\bar{V}^0 \tag{14}$$

A similar relation exists among the currents in any unbalanced three-phase system. For example,

$$\bar{I}_1 = \bar{I}^0 + \bar{I}_1^+ + \bar{I}_1^- \tag{15}$$

$$\bar{I}_2 = \bar{I}^0 + \bar{I}_2^+ + \bar{I}_2^- \tag{16}$$

$$\bar{I}_3 = \bar{I}^0 + \bar{I}_3^+ + \bar{I}_3^- \tag{17}$$

$$\bar{I}_1 + \bar{I}_2 + \bar{I}_3 = 3\bar{I}^0 \tag{18}$$

Figure 117 shows the positive-phase, the negative-phase and also the zero-phase components of the currents in an unbalanced Y-connected, three-phase load with a neutral connection.

The positive-phase components are shown by heavy solid lines. Heavy dotted lines are used for the negative-phase components. The zero-phase components are shown by solid lines. Dot-and-dash lines are used for the original vectors.

The three components, *i.e.*, the positive-phase, the negative-phase and the zero-phase components, of each of the original vectors of an unbalanced three-phase system of voltages or currents may be determined either analytically or graphically from the known magnitudes and phase relations of the original three-phase vectors.

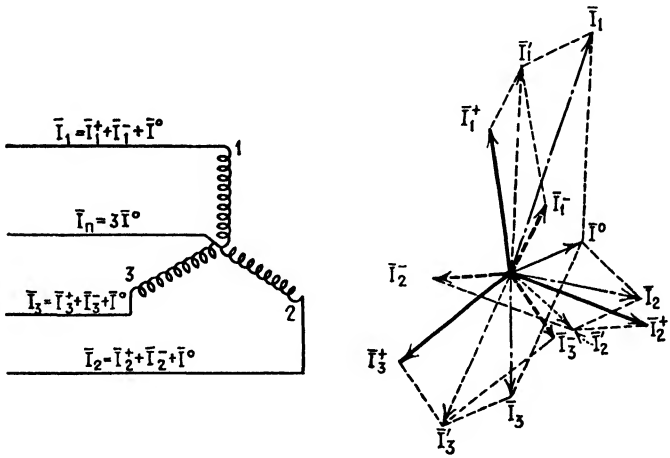


FIG. 117.

Determination of the Positive-phase, the Negative-phase and the Zero-phase Components of a Three-phase System.—Consider the three voltage vectors given by equations (11), (12) and (13). It has already been shown that the zero-phase components are

$$\bar{V}^0 = \frac{\bar{V}_1 + \bar{V}_2 + \bar{V}_3}{3} \tag{19}$$

Also,

$$\bar{V}_1^+ + \bar{V}_2^+ + \bar{V}_3^+ = 0 \tag{20}$$
$$\bar{V}_1^- + \bar{V}_2^- + \bar{V}_3^- = 0 \tag{21}$$

Let $\bar{V}$'s with primes indicate the vectors obtained by subtracting the zero-phase components from each of the given three-phase

vectors. Then, if the phase order of the positive-phase components is clockwise,

$$\bar{V}_1' = \bar{V}_1 - \bar{V}^0 = \bar{V}_1^+ + \bar{V}_1^- = \bar{V}_1^+\underline{|-0^\circ} + \bar{V}_1^-\underline{|+0^\circ} \qquad (22)$$
$$\bar{V}_2' = \bar{V}_2 - \bar{V}^0 = \bar{V}_2^+ + \bar{V}_2^- = \bar{V}_1^+\underline{|-120^\circ} + \bar{V}_1^-\underline{|+120^\circ} \qquad (23)$$
$$\bar{V}_3' = \bar{V}_3 - \bar{V}^0 = \bar{V}_3^+ + \bar{V}_3^- = \bar{V}_1^+\underline{|-240^\circ} + \bar{V}_1^-\underline{|+240^\circ} \qquad (24)$$

If the system is balanced, each of the zero-phase components, $\bar{V}^0$, and each of the negative-phase components is zero.

Let each of the vectors in equation (23) be rotated in a clockwise direction through 120 degrees by applying the operator $\underline{|-120^\circ}$. Then,

$$\bar{V}_2'\underline{|-120^\circ} = \bar{V}_1^+\underline{|-240^\circ} + \bar{V}_1^-\underline{|0^\circ} \qquad (25)$$

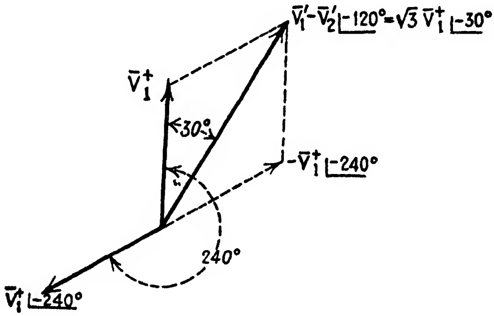


FIG. 118.

Subtracting equation (25) from equation (22) gives

$$(\bar{V}_1' - \bar{V}_2'\underline{|-120^\circ}) = (\bar{V}_1^+\underline{|0^\circ} - \bar{V}_1^+\underline{|-240^\circ})$$
$$+ (\bar{V}_1^-\underline{|0^\circ} - \bar{V}_1^-\underline{|0^\circ})$$
$$= (\bar{V}_1^+\underline{|0^\circ} - \bar{V}_1^+\underline{|-240^\circ}) \qquad (26)$$

The voltages $(\bar{V}_1' - \bar{V}_2'\underline{|-120^\circ})$, $\bar{V}_1^+\underline{|0^\circ}$ and $\bar{V}_1^+\underline{|-240^\circ}$ are shown in Fig. 118.

By referring to Fig. 118 it is seen that

$$(\bar{V}_1' - \bar{V}_2'\underline{|-120^\circ}) = (\bar{V}_1' + \bar{V}_2'\underline{|+60^\circ}) = \sqrt{3}\,\bar{V}_1^+\underline{|-30^\circ} \qquad (27)$$

Hence,

$$\bar{V}_1^+ = \frac{1}{\sqrt{3}}(\bar{V}_1' + \bar{V}_2'\underline{|+60^\circ})\ \underline{|+30^\circ} \qquad (28)$$

$$= \frac{1}{\sqrt{3}}(\bar{V}_1'\underline{|+30^\circ} + \bar{V}_2'\underline{|+90^\circ}) \tag{29}$$

Since $\bar{V}_1' = \bar{V}_1^+ + \bar{V}_1^-$ [equation (22)]

$$\bar{V}_1^- = \bar{V}_1' - \bar{V}_1^+$$

$$= \bar{V}_1' - \frac{\bar{V}_1'}{\sqrt{3}}\underline{|+30^\circ} - \frac{\bar{V}_2'}{\sqrt{3}}\underline{|+90^\circ} \tag{30}$$

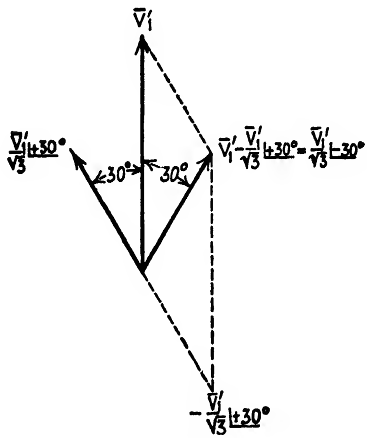


FIG. 119.

Referring to Fig. 119 it is evident that equation (30) reduces to

$$\bar{V}_1^- = \frac{1}{\sqrt{3}}(\bar{V}_1'\underline{|-30^\circ} - \bar{V}_2'\underline{|+90^\circ})$$

$$= \frac{1}{\sqrt{3}}(\bar{V}_1' - \bar{V}_2'\underline{|+120^\circ})\underline{|-30^\circ}$$

$$= \frac{1}{\sqrt{3}}(\bar{V}_1' + \bar{V}_2'\underline{|-60^\circ})\underline{|-30^\circ} \tag{31}$$

$$= \frac{1}{\sqrt{3}}(\bar{V}_1'\underline{|-30^\circ} + \bar{V}_2'\underline{|-90^\circ}) \tag{32}^*$$

Having determined the positive-phase and the negative-phase components $\bar{V}_1^+$ and $\bar{V}_1^-$ for phase 1, the positive-phase and

* Equations (29) and (32) hold only when $\bar{V}_1$ leads $\bar{V}_2$. When $\bar{V}_1$ lags $\bar{V}_2$, equation (32) gives the positive-phase component and equation (29) gives the negative-phase component.

the negative-phase systems of components may be found by applying the proper operators to $\bar{V}_1^+$ and $\bar{V}_1^-$.

$$\bar{V}_1^+ = \bar{V}_1^+ \underline{\vert -0^\circ} \tag{33}$$
$$\bar{V}_2^+ = \bar{V}_1^+ \underline{\vert -120^\circ} \tag{34}$$
$$\bar{V}_3^+ = \bar{V}_1^+ \underline{\vert -240^\circ} \tag{35}$$
$$\bar{V}_1^- = \bar{V}_1^- \underline{\vert +0^\circ} \tag{36}$$
$$\bar{V}_2^- = \bar{V}_1^- \underline{\vert +120^\circ} \tag{37}$$
$$\bar{V}_3^- = \bar{V}_1^- \underline{\vert +240^\circ} \tag{38}$$

From equation (28) it is obvious that to find the magnitude of the positive-phase component $\bar{V}_1^+$ for phase 1, $\bar{V}_2'$ must be

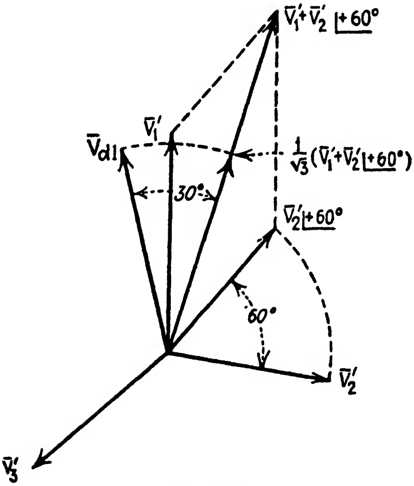


Fig. 120.

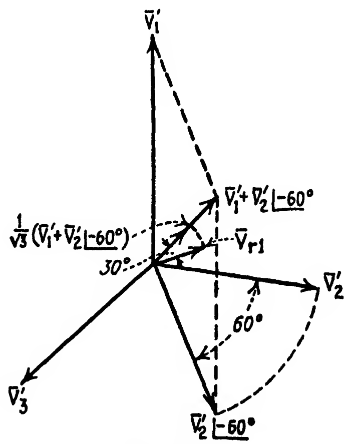


Fig. 121.

rotated in a positive or counter-clockwise direction through 60 degrees, added to $\bar{V}_1'$ and the result divided by the square root of three. The correct phase position of this positive-phase component is found by rotating it, as just found, through 30 degrees in a positive or counter-clockwise direction.

From equation (31) it is obvious that to find the magnitude of the negative-phase component $\bar{V}_1^-$ for phase 1, $\bar{V}_2'$ must be rotated in a negative or clockwise direction through 60 degrees, added to $\bar{V}_1'$ and the result divided by the square root

of three. The correct phase position of this negative-phase component is found by rotating it, as just found, through 30 degrees in a negative or clockwise direction.

The method of finding the positive-phase and the negative-phase components for phase 1 is illustrated in Figs. 120 and 121. A graphical determination of the positive-phase and the negative-phase components is sufficiently accurate for many purposes. When the complex expressions for the three vectors which are to be resolved into positive-phase and negative-phase components are known, an analytical solution is simple and requires little if any more time than the graphical solution.

A Simple Graphical Construction for Finding the Positive-phase and the Negative-phase Components of a Three-phase Circuit Whose Vectors Are Sinusoidal and Contain No Zero-phase Components.—When there are no zero-phase components in the currents or voltages of a three-phase circuit, whose currents and voltages are sinusoidal, and only the magnitudes of the currents or voltages are known, the positive-phase and the negative-phase components may be found by a very simple graphical construction.[1] The vector sum of the line voltages of a three-phase circuit is zero. They therefore can contain no zero-phase components. Also, the vector sum of the line currents of a three-phase, Δ-connected circuit, or of a three-phase, Y-connected circuit without a neutral, must be zero. They can contain no zero-phase components.

Suppose the magnitudes of the line voltages of a three-phase circuit, whose voltages are sinusoidal, are known. Since their vector sum must be zero, they must form the sides of a closed triangle, as shown in Fig. 122, where $\bar{V}_{12}'$, $\bar{V}_{23}'$ and $\bar{V}_{31}'$ are the voltages. Construct an equilateral triangle 2–4–3 on $\bar{V}_{23}'$ as a base. The side 2–4 of this triangle is $\bar{V}_{23}'$ rotated through 60 degrees in a positive direction. From equation (28), page 379, the diagonal 1–4 = $\bar{V}_{14}'$ divided by the square root of three is equal to the magnitude of the positive-phase component of phase 1–2.

Draw the isosceles triangle 1–5–4 on 1–4 as a base, with 30-degree angles at 1 and 4, as shown in Fig. 122. Then,

[1] W. V. Lyon, Unbalanced Three-phase Circuits, Electrical World, June 5, 1920.

$$1\text{–}5 = \bar{V}_{15}{}' = \frac{\bar{V}_{14}{}'}{\sqrt{3}}\underline{|+30^\circ} = \frac{1}{\sqrt{3}}(\bar{V}_{12}{}' + \bar{V}_{23}{}'\underline{|+60^\circ})\underline{|+30^\circ}$$
$$= \bar{V}_{12}^{+} = \text{positive-phase component for phase 1–2}$$

A similar construction, shown in Fig. 123, gives the negative-phase component, $\bar{V}_{12}^{-}$ of phase 1–2.

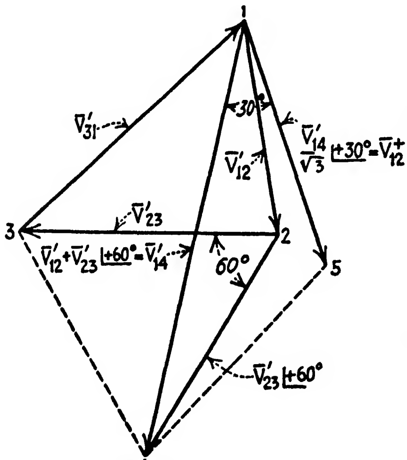


Fig. 122.

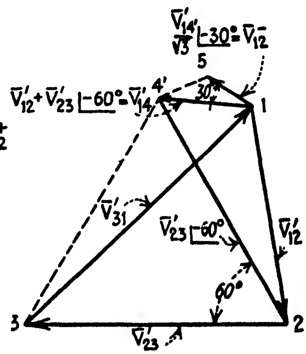


Fig. 123.

Example of the Calculation of the Positive-phase, the Negative-phase and the Zero-phase Components of the Currents in an Unbalanced Y-connected, Three-phase Circuit Whose Currents Are Sinusoidal.—Measurements show that the line currents in an unbalanced Y-connected, three-phase circuit, with neutral connection, whose currents are sinusoidal, are

$$\bar{I}_{01} = 100\underline{|-0^\circ} \text{ amperes}$$
$$\bar{I}_{02} = 100\underline{|-100^\circ} \text{ amperes}$$
$$\bar{I}_{03} = 75\underline{|-250^\circ} \text{ amperes}$$

What are the positive-phase, the negative-phase and the zero-phase components of the currents?

$$\bar{I}_{01} = \bar{I}_1 = 100\underline{|-0^\circ} = 100\{\cos(-0^\circ) + j\sin(-0^\circ)\}$$
$$= 100 + j0$$

$$\bar{I}_{02} = \bar{I}_2 = 100\underline{|-100^\circ} = 100\{\cos(-100^\circ) + j\sin(-100^\circ)\}$$
$$= -17.36 - j98.48$$
$$\bar{I}_{03} = \bar{I}_3 = 75\underline{|-250^\circ} = 75\{\cos(-250^\circ) + j\sin(-250^\circ)\}$$
$$= -25.65 + j70.48$$

$$\bar{I}^0 = \frac{\bar{I}_n}{3} = \frac{\bar{I}_1 + \bar{I}_2 + \bar{I}_3}{3}$$
$$= \frac{(100 - 17.36 - 25.65) + j(0 - 98.48 + 70.48)}{3}$$
$$= 19.00 - j9.33$$
$$\bar{I}_1' = \bar{I}_1^+ + \bar{I}_1^- = \bar{I}_1 - \bar{I}^0 = (100 - j0) - (19.00 - j9.33)$$
$$= 81.00 + j9.33$$
$$\bar{I}_2' = \bar{I}_2^+ + \bar{I}_2^- = \bar{I}_2 - \bar{I}^0 = (-17.36 - j98.48) - (19.00 - j9.33)$$
$$= -36.36 - j89.15$$
$$\bar{I}_3' = \bar{I}_3^+ + \bar{I}_3^- = \bar{I}_3 - \bar{I}^0 = (-25.65 + j70.48) - (19.00 - j9.33)$$
$$= -44.65 + j\,79.81$$

It is obvious from equation (29), page 380, that

$$\bar{I}_1^+ = \frac{1}{\sqrt{3}}\{\bar{I}_1'(\cos 30^\circ + j\sin 30^\circ) + \bar{I}_2'(\cos 90^\circ + j\sin 90^\circ)\}$$
$$= \frac{1}{\sqrt{3}}\{(81.00 + j9.33)(0.866 + j\,0.500) + (-36.36 - j89.15)(0.000 + j1.000)\}$$
$$= \frac{1}{\sqrt{3}}\{(65.48 + j48.58) + (89.15 - j36.36)\}$$
$$= 89.3 + j7.05$$

From equation (32), page 380,

$$\bar{I}_1^- = \frac{1}{\sqrt{3}}\{\bar{I}_1'(\cos 30^\circ - j\sin 30^\circ) + \bar{I}_2'(\cos 90^\circ - j\sin 90^\circ)\}$$
$$= \frac{1}{\sqrt{3}}\{(81.00 + j9.33)(0.866 - j0.500) + (-36.36 - j89.15)(0.000 - j1.000)\}$$
$$= \frac{1}{\sqrt{3}}\{(74.81 - j32.42) + (-89.15 + j36.36)\}$$
$$= -8.28 + j2.27$$

Then,

$$\bar{I}_1^0 = 19.00 - j9.33$$
$$\bar{I}_2^0 = 19.00 - j9.33$$
$$\bar{I}_3^0 = 19.00 - j9.33$$

$$\begin{aligned}
\bar{I}_1^+ &= 89.3 + j7.05 \\
\bar{I}_2^+ &= (89.3 + j7.05)(\cos 120° - j \sin 120°) \\
&= -38.55 - j80.86 \\
\bar{I}_3^+ &= (89.3 + j7.05)\ (\cos 240° - j \sin 240°) \\
&= -50.76 + j73.81 \\
\bar{I}_1^- &= -8.28 + j2.27 \\
\bar{I}_2^- &= (-8.28 + j2.27)(\cos 120° + j \sin 120°) \\
&= 2.174 - j8.305 \\
\bar{I}_3^- &= (-8.28 + j2.27)(\cos 240° + j \sin 240°) \\
&= 6.106 + j6.035
\end{aligned}$$

Another Method of Resolving Voltages or Currents of an Unbalanced Three-phase Circuit into Their Positive-phase, Negative-phase and Zero-phase Components by the Use of an Operator "a" Which Produces a Rotation of 120 Degrees.—Let

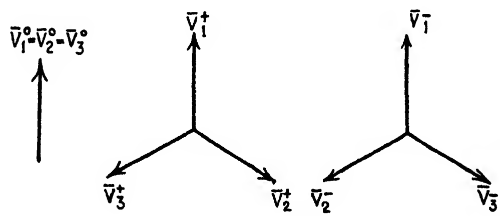


FIG. 124.

the phase voltages of an unbalanced three-phase circuit be $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$. Let $\bar{V}_1$ lead $\bar{V}_2$. Let the operator a produce a rotation of 120 degrees in a positive or counter-clockwise direction. Then,

$$a\ = 1\underline{|+120°} = -0.5 + j0.866$$
$$a^2 = 1\underline{|-120°} = -0.5 - j0.866$$

From equation (19), page 378,

$$\bar{V}^0 = \frac{1}{3}(\bar{V}_1 + \bar{V}_2 + \bar{V}_3) \tag{39}$$

Let the three groups of vectors shown in Fig. 124 represent the phase order of the positive-phase, the negative-phase and the

zero-phase components of the voltages. The three component voltages for phase 1 are not necessarily in phase as shown in Fig. 124.

$$\begin{aligned}(\bar{V}_1 + a\bar{V}_2 + a^2\bar{V}_3) &= (\bar{V}_1^+ + \bar{V}_1^- + \bar{V}_1^0) + a(\bar{V}_2^+ + \bar{V}_2^- + \bar{V}_2^0) \\ &\quad + a^2(\bar{V}_3^+ + \bar{V}_3^- + \bar{V}_3^0) \\ &= (\bar{V}_1^+ + a\,\bar{V}_2^+ + a^2\bar{V}_3^+) + (\bar{V}_1^- + a\bar{V}_2^- + \\ &\quad a^2\bar{V}_3^-) + (\bar{V}_1^0 + a\bar{V}_2^0 + a^2\bar{V}_3^0) \qquad (40)\end{aligned}$$

Since $a = 1\underline{|120°}$ and $a^2 = 1\underline{|240°}$, it is evident, by referring to Fig. 124, that each of the terms $(\bar{V}_1^- + a\bar{V}_2^- + a^2\bar{V}_3^-)$ and $(\bar{V}_1^0 + a\bar{V}_2^0 + a^2\bar{V}_3^0)$ in equation (40) is equal to zero, since it consists of three equal vectors which differ in phase by 120 degrees. The term $(\bar{V}_1^+ + a\bar{V}_2^+ + a^2\bar{V}_3^+)$ consists of three terms which are all in phase and is therefore equal to $3\bar{V}_1^+$. Therefore,

$$\bar{V}_1^+ = \frac{1}{3}(\bar{V}_1 + a\bar{V}_2 + a^2\bar{V}_3) = \frac{1}{3}(\bar{V}_1 + \bar{V}_2\underline{|120°} + \bar{V}_3\underline{|240°}) \qquad (41)$$

It may be shown in a similar way that

$$\bar{V}_1^- = \frac{1}{3}(\bar{V}_1 + a^2\bar{V}_2 + a\bar{V}_3) = \frac{1}{3}(\bar{V}_1 + \bar{V}_2\underline{|-120°} + \bar{V}_3\underline{|-240°}) \qquad (42)$$

If $\bar{V}_1$ lags $\bar{V}_2$ instead of leading it as was assumed, the operator a must be considered to produce a rotation of 120 degrees in a negative or clockwise direction in order to make equations (41) and (42) correct.

Unbalance Factors.—When a circuit is unbalanced, it is often convenient to express the degree of unbalance in terms of factors. Since a balanced circuit has only positive-sequence components in its voltages and currents, the degree of unbalance is expressed in terms of these. There are two unbalance factors for the voltage of a three-phase circuit. These are:

$$\text{Negative-sequence unbalance factor} = \frac{V^-}{V^+} \qquad (43)$$

and

$$\text{Zero-sequence unbalance factor} = \frac{V^0}{V^+} \qquad (44)$$

The unbalance factors for the currents are similarly defined. In general, the unbalance factors for the voltages and the currents are different. In many cases the factors for the zero-sequence components are zero, since these components are frequently absent in practice.

Mutual Induction between a Three-phase Transmission Line and a Neighboring Telephone Line.—The vector sum of the positive-phase components and the vector sum of the negative-phase components of the currents of a transmission line are each zero. Therefore, if the conductors of a three-phase transmission line could be equidistant from the conductors of a telephone line, which runs parallel to the transmission line, these component currents could produce no inductive effects on the telephone line. Although it is impossible to have the conductors of a transmission line equidistant from the conductors of a telephone line, they can, on the average, be made equidistant by properly transposing them. By proper transposition, the inductive effects of the positive-phase and negative-phase components may be made zero. This is not true of the zero-phase components. These are all in phase and no amount of transposition of the conductors of the transmission line would alter their inductive effects on the telephone line. To get rid of the inductive effects of the zero-phase components, the telephone line must be transposed. The zero-phase components or *residuals,* as they are called by telephone engineers, play a very important part in the interference effects produced on telephone lines by unbalanced transmission lines which are operated with grounded neutrals.

The whole analysis of the interference between transmission lines and neighboring telephone lines is much simplified by resolving the currents in the transmission line into positive-phase, negative-phase and zero-phase components.

Power in an Unbalanced Three-phase Circuit When the Currents and Voltages Are Sinusoidal.—The power in a three-phase circuit is

$$P_0 = P_1 + P_2 + P_3 \tag{45}$$

where P_1, P_2 and P_3 are the powers in phases 1, 2 and 3, respectively.

$$P_1 = V_1 I_1 \cos \theta_1 \tag{46}$$

$$P_2 = V_2 I_2 \cos \theta_2 \tag{47}$$

$$P_3 = V_3 I_3 \cos \theta_3 \tag{48}$$

where the V's, I's and θ's represent phase values of voltage, current and power-factor angle.

If V_1 and I_1 are each resolved into positive-phase, negative-phase and zero-phase components, the expression for P_1 becomes

$$\begin{aligned} P_1 = V^+I^+ \cos \theta^{V_1^+}_{I_1^+} &+ V^-I^- \cos \theta^{V_1^-}_{I_1^-} + V^0I^0 \cos \theta^{V_1^0}_{I_1^0} \\ &+ V^+I^- \cos \theta^{V_1^+}_{I_1^-} + V^+I^0 \cos \theta^{V_1^+}_{I_1^0} \\ &+ V^-I^+ \cos \theta^{V_1^-}_{I_1^+} + V^-I^0 \cos \theta^{V_1^-}_{I_1^0} \\ &+ V^0I^+ \cos \theta^{V_1^0}_{I_1^+} + V^0I^- \cos \theta^{V_1^0}_{I_1^-} \end{aligned} \tag{49}$$

where V^+, I^+, V^-, $I,^-$ V^0 and I^0 represent the numerical values of the positive-phase, the negative-phase and the zero-phase components of the voltages and currents. The limits on the phase angles θ indicate to which sequence they refer.

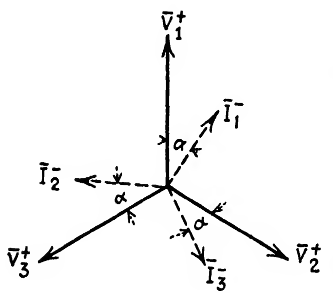


Fig. 125.

Expressions which are similar to equation (49) may be written for P_2 and P_3.

When the expressions for P_1, P_2 and P_3 are added to give the total power P_0, it is evident that the sums of the terms involving unlike components are zero. For example, the sum of the three following terms is zero:

$$\begin{aligned} V^+I^- \cos \theta^{V_1^+}_{I_1^-} + V^+I^- \cos \theta^{V_2^+}_{I_2^-} + V^+I^- \cos \theta^{V_3^+}_{I_3^-} &= \\ V^+I^- \left(\cos \theta^{V_1^+}_{I_1^-} + \cos \theta^{V_2^+}_{I_2^-} + \cos \theta^{V_3^+}_{I_3^-}\right) &= 0 \end{aligned} \tag{50}$$

The components in equation (50) are plotted in Fig. 125.

By referring to Fig. 125, it is evident that equation (50) may be written in the form:

$$\begin{aligned} V^+I^-\{\cos \alpha + \cos (120° + \alpha) + \cos (120° - \alpha)\} &= \\ V^+I^-\{\cos \alpha + \cos 120° \cos \alpha - \sin 120° \sin \alpha & \\ + \cos 120° \cos \alpha + \sin 120° \sin \alpha\} &= \\ V^+I^-\left\{\cos \alpha - \frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha - \frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha\right\} &= 0 \end{aligned}$$

The sum of the other terms involving components which are unlike may similarly be shown to be zero. Therefore,

$$P_0 = 3V^+I^+ \cos \theta_{I^+}^{V^+} + 3V^-I^- \cos \theta_{I^-}^{V^-} + 3V^0I^0 \cos \theta_{I^0}^{V^0} \quad (51)$$

The total average power in an unbalanced three-phase circuit, which has no zero-phase components, is equal to the sum of the powers developed by the positive-phase and negative-phase components. When negative-phase components are present in the currents and voltages of a three-phase circuit, *i.e.*, when the circuit is unbalanced, the power developed in at least one phase is greater, and in at least one other phase is less than the average power per phase. If there were no negative-phase components, the power developed by all phases would be the same. The effect of the negative-phase components is to transfer power from one phase to another. The principle of the phase balancer depends on this transfer of power from one phase to another by negative-phase components.

It should be noted that the average power due to the positive-phase currents and the negative-phase or the zero-phase voltages is zero.

Phase Balancer.—It is necessary, in certain cases, for central stations to supply large amounts of single-phase power for special purposes, such as for the operation of electric railways using single-phase, alternating-current series motors. This single-phase load not only badly unbalances the voltage of the system supplying it but also very much decreases the permissible output from the generating equipment. The operation of certain types of polyphase apparatus, notably the synchronous converter, is difficult on circuits whose voltage is out of balance.

The impedance of a polyphase rotating machine, such as a synchronous motor provided with a damping winding, or an induction motor, is much less for the negative-phase components of an unbalanced system of voltages than for the positive-phase components. Such a machine, when connected to a circuit having unbalanced voltages, takes negative-phase currents which are large compared with the positive-phase currents. The negative-phase currents are substantially opposite in phase to the negative-phase currents in the load which causes the unbalanced voltages. For this reason, a synchronous motor with

damping winding or an induction motor partially restores the condition of balance of currents in an unbalanced circuit to which it is connected, but such a machine alone cannot restore the condition of complete balance of currents.

A type of apparatus has been developed by means of which complete balance may be restored to an unbalanced three-phase circuit. This is accomplished by taking from the unbalanced circuit negative-phase currents equal in magnitude but opposite in phase to those caused by the existing unbalanced load. Since unbalanced voltages are usually due to unbalanced line drops caused by an unbalanced load, balancing the currents of a circuit usually balances the voltages.

The *phase balancer*, as the machine is called, consists of a synchronous motor, with a low-impedance damper, driving a three-phase synchronous alternator. One phase of the alter-

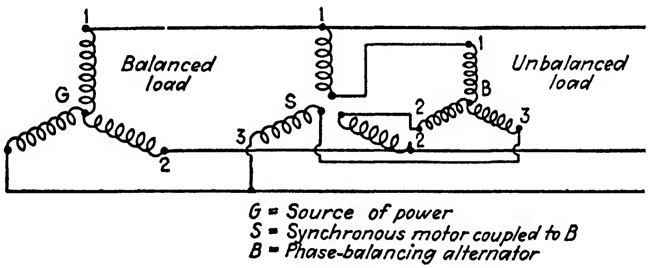


FIG. 126.

nator is connected in series with each phase of the synchronous motor, but the phase orders of the alternator and motor are opposite. The alternator impresses negative-phase voltages of such magnitude and phase that the motor takes negative-phase currents from the line which are equal and opposite to those in the unbalanced load. An automatic device is used to adjust the magnitude and phase of negative-phase voltages of the phase-balancing alternator. The magnitude of the voltages is adjusted by means of the field excitation of the alternator. Their phase is varied by rotating the magnetic axis of the field with respect to the field structure by the use of two independently excited field windings displaced 90 degrees from each other on the field structure.

The development of the phase balancer would have been improbable without the knowledge that an unbalanced three-

phase load, with sinusoidal currents and voltages and without a neutral connection, could be resolved into two balanced three-phase loads having opposite phase orders. The schematic diagram of connections for a phase balancer to balance currents is shown in Fig. 126.

Copper Loss in an Unbalanced Three-phase Circuit in Terms of the Positive-phase, the Negative-phase and the Zero-phase Components of the Currents.—The copper loss in phase 1 is

$$P_{1\ copper} = (I_{1\ phase})^2 \times r_1$$

The positive-phase, the negative-phase and the zero-phase components of the currents are shown in Fig. 127.

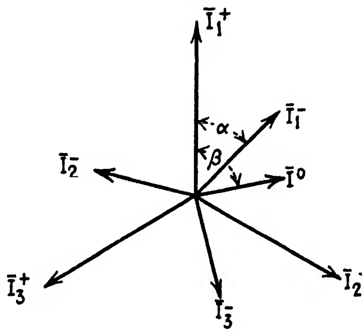


FIG. 127.

If $\bar{I}_1^-$ and $\bar{I}^0$ are each resolved into two quadrature components with respect to $\bar{I}_1^+$, it is evident, by referring to Fig. 127, that

$$\begin{aligned} P_{1\ copper} &= [(I_1^+ + I_1^- \cos\alpha + I^0 \cos\beta)^2 \\ &\qquad + (I_1^- \sin\alpha + I^0 \sin\beta)^2]r_1 \\ &= [(I_1^+)^2 + (I_1^-)^2 + (I^0)^2]r_1 + 2[I_1^+ I_2^- \cos\alpha + I_1^+ I^0 \cos\beta \\ &\qquad + I_1^- I^0 \cos\alpha\cos\beta + I_1^- I^0 \sin\alpha\sin\beta]r_1 \qquad (52) \end{aligned}$$

Similarly, resolving $\bar{I}_2^-$ and $\bar{I}^0$ each into two quadrature components with respect to $\bar{I}_2^+$ gives

$$\begin{aligned} P_{2\ copper} &= (I_{2\ phase})^2 \times r_2 \\ &= \{[I_2^+ + I_2^- \cos(240^\circ - \alpha) + I^0 \cos(120^\circ - \beta)]^2 \\ &\qquad + [I_2^- \sin(240^\circ - \alpha) + I^0 \sin(120^\circ - \beta)]^2\}r_2 \\ &= [(I_2^+)^2 + (I_2^-)^2 + (I^0)^2]r_2 + 2[I_2^+ I_2^- \cos(240^\circ - \alpha) \\ &\qquad + I_2^+ I^0 \cos(120^\circ - \beta) \\ &\qquad\quad + I_2^- I^0 \cos(240^\circ - \alpha)\cos(120^\circ - \beta) \\ &\qquad\qquad + I_2^- I^0 \sin(240^\circ - \alpha)\sin(120^\circ - \beta)]r_2 \qquad (53) \end{aligned}$$

Resolving $\bar{I}_3^-$ and $\bar{I}^0$ each into quadrature components with respect to $\bar{I}_3^+$ gives

$$\begin{aligned} P_{3\ copper} &= (I_{3\ copper})^2 \times r_3 \\ &= \{[I_3^+ + I_3^- \cos(120^0 - \alpha) + I^0 \cos(240^\circ - \beta)]^2 \\ &\qquad + [I_3^- \sin(120^\circ - \alpha) + I^0 \sin(240^\circ - \beta)]^2\}r_3 \end{aligned}$$

$$= [(I_3^+)^2 + (I_3^-)^2 + (I^0)^2]r_3 + 2[I_3^+I_3^- \cos (120° - \alpha)$$
$$+ I_3^+I^0 \cos (240° - \beta)$$
$$+ I_3^-I^0 \cos (120° - \alpha) \cos (240° - \beta)$$
$$+ I_3^-I^0 \sin (120° - \alpha) \sin (240° - \beta)]r_3 \quad (54)$$

Since $I_1^+ = I_2^+ = I_3^+$ and $I_1^- = I_2^- = I_3^-$ in magnitude, the subscripts, 1, 2 and 3 may be omitted, since the currents and voltages enter only in magnitude in equations (52), (53) and (54).

When the resistances r_1, r_2 and r_3 are equal, as would be the case for a motor or a generator, the equation for the total power due to copper loss becomes

$$P_0 = P_1 + P_2 + P_3$$
$$= 3\{(I^+)^2 + (I^-)^2 + (I^0)^2\}r$$
$$+ 2\{I^+I^-[\cos \alpha + \cos (120° - \alpha) + \cos (240° - \alpha)]$$
$$+ I^+I^0[\cos \beta + \cos (120° - \beta) + \cos (240° - \beta)]$$
$$+ I^-I^0[\cos \alpha \cos \beta + \cos (240° - \alpha) \cos (120° - \beta)$$
$$+ \cos (120° - \alpha) \cos (240° - \beta)]$$
$$+ I^-I^0 [\sin \alpha \sin \beta + \sin (240° - \alpha) \sin (120° - \beta)$$
$$+ \sin (120 - \alpha) \sin (240° - \beta)]\}r \quad (55)$$

The sum of the cosines of any three angles which differ by 120 degrees is zero. All terms on the right-hand side of equation (55) except the first are, therefore, equal to zero. These terms may be written in the following form:

$$I^-I^0\{\cos [(\alpha - \beta)] + \cos [(\alpha - \beta) - 120°]$$
$$+ \cos [(\alpha - \beta) - 240°]\} = 0$$

The total copper loss, when the resistances of all three phases are equal, as they would be in the case of motors or generators, is equal to

$$P_0 = 3[(I^+)^2 + (I^-)^2 + (I^0)^2]r \quad (56)$$

That is, the total copper loss in an unbalanced three-phase system whose phase resistances are equal, is equal to three times the sum of the squares of the positive-phase, negative-phase and zero-phase components of the currents multiplied by the phase resistance.

Positive-phase, Negative-phase and Zero-phase Impedances of a Three-phase Circuit.—Except when the impedances of the three phases are identical, the positive-phase, negative-phase and zero-phase voltage drops in any phase depend on all three

components of the current. Assume wye connection. Let the impedances of the phases be $\bar{z}_1$, $\bar{z}_2$ and $\bar{z}_3$ and let the phase voltages and phase currents be respectively $\bar{V}_{01}$, $\bar{V}_{02}$ and $\bar{V}_{03}$ and $\bar{I}_{01}$, $\bar{I}_{02}$ and $\bar{I}_{03}$. Then,

$$\bar{V}_{01} = \bar{I}_{01}\bar{z}_1 \tag{57}$$
$$\bar{V}_{02} = \bar{I}_{02}\bar{z}_2 \tag{58}$$
$$\bar{V}_{03} = \bar{I}_{03}\bar{z}_3 \tag{59}$$

Let $\bar{V}_{01}$ lead $\bar{V}_{02}$. Then, from equations (39), (41) and (42),

$$\bar{V}^0 = \frac{1}{3}(\bar{V}_{01} + \bar{V}_{02} + \bar{V}_{03}) \tag{60}$$

$$\bar{V}_1^+ = \frac{1}{3}(\bar{V}_{01} + a\bar{V}_{02} + a^2\bar{V}_{03}) \tag{61}$$

$$\bar{V}_1^- = \frac{1}{3}(\bar{V}_{01} + a^2\bar{V}_{02} + a\bar{V}_{03}) \tag{62}$$

where a has the same significance as on page 385, *i.e.*, it is an operator which rotates the vector to which it is applied through 120 degrees in a positive or counter-clockwise direction. If $\bar{V}_{01}$ lags $\bar{V}_{02}$, a must produce a rotation of 120 degrees in a negative or clockwise direction in order to make equations (61) and (62) correct.

Substituting the impedance drops from equations (57), (58) and (59) for the voltages in equation (60) gives

$$\begin{aligned}
\bar{V}^0 &= \frac{1}{3}(\bar{I}_{01}\bar{z}_1 + \bar{I}_{02}\bar{z}_2 + \bar{I}_{03}\bar{z}_3) \\
&= \frac{1}{3}\{(\bar{I}_1^+ + \bar{I}_1^- + \bar{I}_1^0)\bar{z}_1 + (\bar{I}_2^+ + \bar{I}_2^- + \bar{I}_2^0)\bar{z}_2 \\
&\qquad + (\bar{I}_3^+ + \bar{I}_3^- + \bar{I}_3^0)\bar{z}_3\} \\
&\ \frac{1}{3}\{(\bar{I}_1^+ + \bar{I}_1^- + \bar{I}_1^0)\bar{z}_1 + (a^2\bar{I}_1^+ + a\bar{I}_1^- + \bar{I}_1^0)\bar{z}_2 \\
&\qquad + (a\bar{I}_1^+ + a^2\bar{I}_1^- + \bar{I}_1^0)\bar{z}_3\} \\
&= \bar{I}_1^+\frac{1}{3}(\bar{z}_1 + a^2\bar{z}_2 + a\bar{z}_3) + \bar{I}_1^-\frac{1}{3}(\bar{z}_1 + a\bar{z}_2 + a^2\bar{z}_3) \\
&\qquad + \bar{I}^0\frac{1}{3}(\bar{z}_1 + \bar{z}_2 + \bar{z}_3)
\end{aligned} \tag{63}$$

Substituting impedance drops for voltages in equation (61) gives

$$\bar{V}_1^+ = \frac{1}{3}(\bar{I}_{01}\bar{z}_1 + a\bar{I}_{02}\bar{z}_2 + a^2\bar{I}_{03}\bar{z}_3)$$
$$= \frac{1}{3}\{(\bar{I}_1^+ + \bar{I}_1^- + \bar{I}_1^0)\bar{z}_1 + (\bar{I}_2^+ + \bar{I}_2^- + \bar{I}_2^0)a\bar{z}_2 + (\bar{I}_3^+ + \bar{I}_3^- + \bar{I}_3^0)a^2\bar{z}_3\}$$
$$= \frac{1}{3}\{(\bar{I}_1^+ + \bar{I}_1^- + \bar{I}_1^0)\bar{z}_1 + (a^2\bar{I}_1^+ + a\bar{I}_1^- + \bar{I}_1^0)a\bar{z}_2 + (a\bar{I}_1^+ + a^2\bar{I}_1^- + \bar{I}_1^0)a^2\bar{z}_3\} \quad (64)$$

Since in the above equations a is an operator which rotates through 120 degrees in a positive direction, $a = 1\underline{|120°} = 1\underline{|-240°}$, $a^2 = 1\underline{|240°} = 1\underline{|-120°}$, $a^3 = 1\underline{|360°} = 1\underline{|0°}$ and $a^4 = 1\underline{|480°} = 1\underline{|120°} = 1\underline{|-240°}$. Equation (64) may be written

$$\bar{V}_1^+ = \bar{I}_1^+\frac{1}{3}(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) + \bar{I}_1^-\frac{1}{3}(\bar{z}_1 + a^2\bar{z}_2 + a\bar{z}_3) + \bar{I}_1^0\frac{1}{3}(\bar{z}_1 + a\bar{z}_2 + a^2\bar{z}_3) \quad (65)$$

In a similar way it may be shown that

$$\bar{V}_1^- = \bar{I}_1^+\frac{1}{3}(\bar{z}_1 + a\bar{z}_2 + a^2\bar{z}_3) + \bar{I}_1^-\frac{1}{3}(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) + \bar{I}_1^0\frac{1}{3}(\bar{z}_1 + a^2\bar{z}_2 + a\bar{z}_3) \quad (66)$$

$$\frac{1}{3}(\bar{z}_1 + a\bar{z}_2 + a^2\bar{z}_3) = \frac{1}{3}(\bar{z}_1 + \bar{z}_2\underline{|+120°} + \bar{z}_3\underline{|+240°}) = \bar{z}_1^+ \quad (67)$$
$$\frac{1}{3}(\bar{z}_1 + a^2\bar{z}_2 + a\bar{z}_3) = \frac{1}{3}(\bar{z}_1 + \bar{z}_2\underline{|-120°} + \bar{z}_3\underline{|-240°}) = \bar{z}_1^- \quad (68)$$
$$\frac{1}{3}(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = \bar{z}^0 \quad (69)$$

The quantities $\bar{z}_1^+$, $\bar{z}_1^-$ and $\bar{z}^0$ given by equations (67), (68) and (69) are called the *positive-phase*, the *negative-phase* and the *zero-phase impedances of the circuit for phase* 1. Similar expressions hold for the corresponding impedances of the other phases. Therefore, according to equations (63), (65) and (66),

$$\bar{V}_1^0 = \bar{I}_1^+\bar{z}_1^- + \bar{I}_1^-\bar{z}_1^+ + \bar{I}^0\bar{z}^0 \quad (70)$$
$$\bar{V}_1^+ = \bar{I}_1^+\bar{z}^0 + \bar{I}_1^-\bar{z}_1^- + \bar{I}^0\bar{z}_1^+ \quad (71)$$
$$\bar{V}_1^- = \bar{I}_1^+\bar{z}_1^+ + \bar{I}_1^-\bar{z}^0 + \bar{I}^0\bar{z}_1^- \quad (72)$$

Similar equations hold for the zero-phase, the positive-phase and the negative-phase components of phases 2 and 3.

If the impedances $\bar{z}_1$, $\bar{z}_2$ and $\bar{z}_3$ of the three phases are identical, each of the impedances $\bar{z}_1^+$ and $\bar{z}_1^-$ is equal to zero and $\bar{z}^0 = \bar{z}$ where $\bar{z} = \bar{z}_1 = \bar{z}_2 = \bar{z}_3 =$ the phase impedance. [See equations (67), (68) and (69).]

The positive-phase, negative-phase and zero-phase impedances [equations (67), (68) and (69)] are so called because they are found from the actual phase impedances in exactly the same manner as the positive-phase, negative-phase and zero-phase currents or voltages are found from their actual phase values. They cannot be used with the corresponding components of current to find the positive-phase, negative-phase and zero-phase components of the voltage drops in a circuit, *i.e.*, $I_1^+z_1^+$ never gives V_1^+ and $I_1^-z_1^-$ never gives V_1^-. They must be used in accordance with equations (70), (71) and (72) to get V_1^+, V_1^- and V_1^0.

Another System of Notation for the Symmetrical-phase Components.—The number of components into which the vectors of an unbalanced system are divided depends upon the number of phases. When there are more than three components, other symbols than zero, plus and minus are commonly employed. The symbols now generally used are the subscript zero for the zero-phase components, the subscript 1 for the positive-phase components, the subscript 2 for the negative-phase components and the subscripts 3, 4 etc. for the other components. This system is also in use for the components of a three-phase system. It is particularly convenient when dealing with the zero-sequence, positive-sequence and negative-sequence impedances. When the subscripts 0, 1, 2 etc. are used for the components, the subscripts *a*, *b*, *c* etc. are used to distinguish the phases. Using this notation, equations (70), (71) and (72) become

$$\bar{V}_{a0} = \bar{I}_{a1}\bar{z}_{a2} + \bar{I}_{a2}\bar{z}_{a1} + \bar{I}_{a0}\bar{z}_{a0} \quad (73)$$

$$\bar{V}_{a1} = \bar{I}_{a1}\bar{z}_{a0} + \bar{I}_{a2}\bar{z}_{a2} + \bar{I}_{a0}\bar{z}_{a1} \quad (74)$$

$$\bar{V}_{a2} = \bar{I}_{a1}\bar{z}_{a1} + \bar{I}_{a2}\bar{z}_{a0} + \bar{I}_{a0}\bar{z}_{a2} \quad (75)$$

When the above notation is used, the sum of the subscripts on each of the right-hand members of equations (73), (74) and (75) is equal to the sum of the subscripts on the voltages, if

$2 + 2 = 4$ is considered equal to 1 and $1 + 2 = 3$ is considered equivalent to 0.

Effect of Impressing an Unbalanced Voltage on a Three-phase Alternating-current Motor or Generator.—The resolution of the voltages and currents of three-phase motors and generators, which operate under unbalanced conditions, into positive-phase and negative-phase components (zero-phase components do not ordinarily exist in motors or generators) is one of the most powerful methods of attacking the problems involved in determining the effect of the unbalanced conditions on operation. Equation (56) shows that the positive-phase and negative-phase components of the currents, and also the zero-phase components if they exist, contribute to the total copper loss. The torque developed in a motor or generator by the negative-phase components of the armature currents is opposite to that developed by the positive-phase components and therefore subtracts from the total net torque. The presence of negative components in the armature currents not only increases the total armature copper loss in a machine, but it also subtracts from the total power developed. The torque developed per ampere of the negative-phase components of the armature currents is much smaller than the torque developed per ampere of the positive-phase components of these currents.

If the positive-phase and negative-phase components of the voltages impressed on a circuit are known, the positive-phase and negative-phase component currents may be calculated, provided the resistances, inductances and capacitances of the circuit are constant and their magnitudes are known. In a motor or generator, the apparent impedance for the negative-phase components of the currents is different from the apparent impedance for the positive-phase components. Equal negative-phase and positive-phase components in the impressed voltages produce negative-phase and positive-phase components in the currents of quite different magnitudes.

The study of the operation of the three-phase induction motor under unbalanced conditions of impressed voltage, or even when it is operated single-phase, may be attacked with great advantage from the standpoint of positive-phase and negative-phase components. The study of the single-phase induction motor may be

simplified by considering the single-phase current to be resolved into the positive-phase and negative-phase components of an equivalent three-phase motor operating with one line conductor disconnected.

Symmetrical-phase Components for Four-phase and Six-phase Circuits.—The chief use for symmetrical-phase components is in the solution of three-phase problems which involve unbalanced conditions in power circuits and in power apparatus such as motors and generators. Symmetrical-phase components are particularly useful when the circuit constants are balanced, *i.e.*, when the impedances of the phases are identical. Under these conditions, there is no reaction between the components of the different sequences in any phase. For example, in a three-phase circuit, when the impedances are identical, currents of positive sequence produce only voltages of positive sequence, currents of negative sequence produce only voltages of negative sequence and currents of zero sequence produce only voltages of zero sequence. [See equations (67), (68) and (69) and equations (70), (71) and (72).] Also, under these conditions, the total resistance loss is equal to the sum of the squares of the positive-sequence, the negative-sequence and the zero-sequence currents multiplied by three times the phase resistance [see equation (56)], and there is no average power produced by the currents of one sequence with the voltages of another sequence. [See equation (50).] Similar statements apply to circuits with more than three phases. Although most power is generated three-phase and distributed three-phase, there is considerable demand for systems with more than three phases, chiefly for the operation of synchronous converters and large mercury-arc rectifiers. The number of symmetrical-phase components of the voltages and of the currents of an unbalanced system is equal to the number of phases. There are three components for a three-phase system, four for a four-phase system and six for a six-phase system. All systems contain zero-sequence components, although under certain conditions they may be zero.

The relative phase relations between the component vectors among themselves for the different sequences are shown in Fig. 128 and Fig. 129, respectively, for four-phase and six-phase circuits. These figures are similar to Fig. 124 shown on page 385

for a three-phase circuit. In these figures the letters *a*, *b*, *c* etc. are used to indicate phases. The figures 0, 1, 2 etc. are used for the components. The method of solving for the components

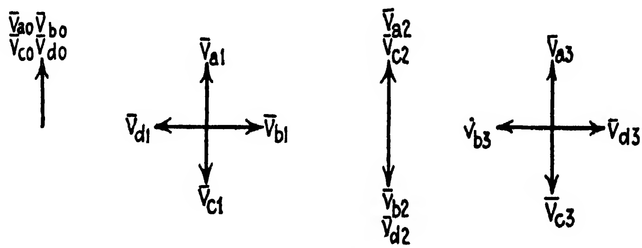


Fig. 128.

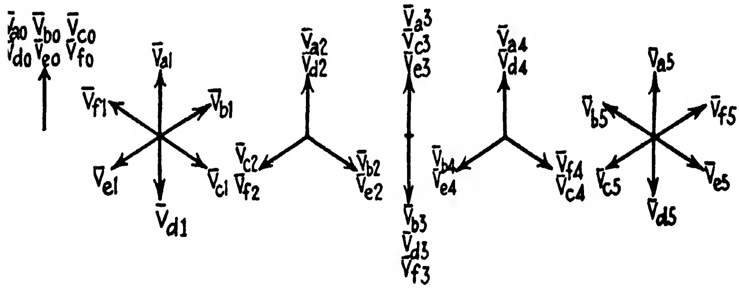


Fig. 129.

of four-phase and six-phase systems is similar to that used for a three-phase system. For the method of solution, see "Symmetrical Components," by Wagner and Evans.

CHAPTER XIV

REACTANCE OF A TRANSMISSION LINE

Reactance of a Single-phase Transmission Line.—When all of the flux of a circuit does not link all of the current, a more fundamental conception of flux linkages than flux linkages with turns or with conductors must be used in determining the self-inductance or mutual inductance of the circuit. Such a case arises when the portion of the self-inductance of a conductor which is due to the flux within the conductor is determined. A similar case occurs in the determination of the portion of the mutual inductance of a circuit which is due to the flux which lies within the conductors. In such cases, it is necessary to consider the flux linkages with respect to current. When all of the flux considered lies without the conductors, or when that portion of the flux within the conductors may be neglected, it makes no difference in the final result whether the linkages are taken with respect to the current or merely with respect to the circuit. Since flux linkages with current must be used in a portion of what follows, they will be used throughout for uniformity. As all currents and fluxes are vectors, they must be considered either vectorially or as instantaneous values. The sense in which they are considered in what follows is clearly indicated by the notation which has been used throughout this book. Small letters indicate instantaneous values. Capital letters with a dash over them indicate vectors.

Consider a single-phase transmission line consisting of two straight, parallel, cylindrical conductors, A and B, of circular cross section, each of r centimeters radius. Let the distance between the axes of the conductors be D centimeters.

The intensity $\mathcal{H}$ of the magnetic field due to a long, straight, cylindrical conductor of circular cross section at a point p outside the conductor and at a perpendicular distance x from its axis is

$$\mathcal{H} = \frac{2i}{x} \text{ gausses} \tag{1}$$

where i is the current in the conductor in abamperes. Equation (1) assumes that the current distribution over the cross section of the conductor is uninfluenced by any magnetic field other than that produced by the current in the conductor itself. The effect is the same as if the current were all concentrated in the axis of the conductor.

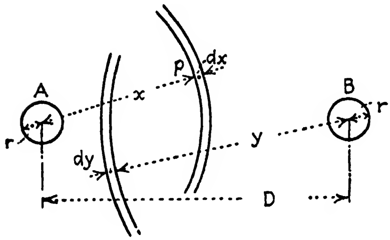


FIG. 130.

For a transmission line carrying current of usual frequency and having the usual spacing between conductors as compared with their diameter, the effect of the current in any conductor on the current distribution in the other conductors may be neglected. This effect cannot be neglected in cables, since the separation between the conductors of cables is small compared with the diameter of the conductors.

The flux density at the point p is

$$\mathcal{B} = \mathcal{H}\mu = \frac{2i}{x}\mu \text{ gausses} \tag{2}$$

$$= \frac{2i}{x} \text{ gausses} \tag{3}$$

since the permeability μ is unity.

Consider the single-phase line shown in Fig. 130.

The total inductance of the conductor A consists of three parts, *viz.:*

(*a*) That due to the flux linkages resulting from the flux produced within the conductor by its own current, *i.e.*, by the current i_A;

(*b*) That due to the flux linkages resulting from that portion of the flux produced by the current i_A, which lies between the surface of the conductor and infinity, *i.e.*, between the limits r and infinity;

(*c*) That due to the flux linkages resulting from so much of the flux produced by the current i_B in conductor B as links conductor A. This, per unit current, is the mutual inductance of B on A.

The parts contributed by (*a*) and (*b*) together, per unit current, constitute the self-inductance of the conductor A.

PART *a*.—Due to skin effect, the current density within the conductors is not uniform, but for ordinary frequencies and sizes of conductors used in power transmission, no great error is produced in the total inductance per conductor by assuming uniform current density. In fact, the part of the inductance due to the flux linkages within the conductor may be neglected without introducing any appreciable error for high-voltage transmission lines with the usual separation between conductors. Low-voltage lines are usually so short that the line inductance ceases to be an important factor. With a ratio of D to r as low as 50, the flux linkages caused by the flux within the conductors account for only about seven per cent of the total line inductance.

Assuming uniform current density within the conductors, the current density at any point within the conductor A is

$$\rho = \frac{i_A}{\pi r^2} \text{ abamperes per square centimeter} \tag{4}$$

The current in that portion of the conductor whose radius is x centimeters is

$$\frac{\pi x^2}{\pi r^2} i_A = \frac{x^2}{r^2} i_A \text{ abamperes} \tag{5}$$

The field intensity at a distance x centimeters from the axis of the conductor due to this portion of the current is

$$\frac{2x^2}{xr^2} i_A = \frac{2x}{r^2} i_A \text{ gausses} \tag{6}$$

The field intensity within a hollow cylindrical conductor due to the current it carries is zero. Hence, the field at a distance x from the axis of a conductor of circular cross section, due to the current in that portion of the conductor without x, is zero. This assumes that the current is either uniformly distributed or has the same density at equal distances from the axis of the conductor.

The flux through an element of the conductor of radius x centimeters, width dx centimeters and length one centimeter is

$$\frac{2x}{r^2} i_A \mu' dx \text{ maxwells} \tag{7}$$

where μ' is the permeability of the conductor.

This flux does not link all of the current in the conductor, but only that portion $\frac{x^2}{r^2}i_A$ which is within the radius x. The total flux linkages with the current per centimeter length of the conductor are

$$\int_0^r \left(\frac{x^2}{r^2}\right)\left(\frac{2x}{r^2}\right)i_A{}^2\mu' dx = \frac{\mu'}{2}i_A{}^2 \tag{8}$$

For non-magnetic conductors, such as are used for transmission lines, this reduces to

$$\frac{i_A{}^2}{2} \tag{9}$$

PART *b*.—The flux density due to the current i_A at any point p outside the conductor A and distant x from the axis of the conductor is (see Fig. 130, page 400)

$$\mathcal{B} = \frac{2i_A}{x} \text{ gausses} \tag{10}$$

The flux through an annulus of radius x centimeters, width dx centimeters and length one centimeter measured parallel to the axis of the conductor, is

$$\mathcal{B}dx = \frac{2i_A}{x}\, dx \text{ maxwells} \tag{11}$$

This links the current i_A in the conductor A. The total flux linkages with the current i_A in conductor A are, therefore,

$$\int_r^\infty \frac{2i_A{}^2}{x}\, dx = 2i_A{}^2 \log_\epsilon \frac{\infty}{r} \tag{12}$$

per centimeter length of conductor A.

PART *c*.—The flux due to the current i_B, in conductor B, which links the current i_A, in conductor A, includes all the flux produced by the current i_B which lies between a distance D from the axis of conductor B and infinity. This assumes that the current i_A in conductor A is concentrated in its axis. The flux linkages with the current i_A due to this flux are (Fig. 130)

$$\int_D^\infty \frac{2i_Ai_B}{y}\, dy = 2i_Ai_B \log_\epsilon \frac{\infty}{D} \tag{13}$$

per centimeter length of conductor A.

The flux due to the current i_B which lies at a distance less than D from the axis of conductor B does not link the current in conductor A and therefore cannot produce mutual inductance on conductor A.

The total resultant flux linkages with the current i_A in conductor A per centimeter length of conductor A are

$$2i_A^2 \log_e \frac{\infty}{r} + \frac{\mu'}{2}i_A^2 + 2i_Ai_B \log_e \frac{\infty}{D} \tag{14}$$

Since a single-phase line is considered, i_A must be equal and opposite to i_B at every instant, no leakage being assumed. Therefore,

$$i_A = -i_B = i$$

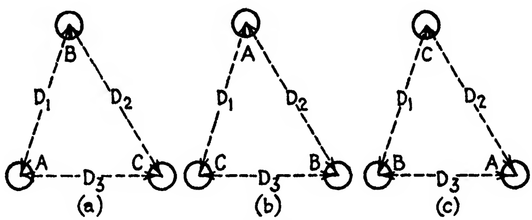


FIG. 131.

and the expression for the resultant linkages becomes

$$2i^2 \log_e \frac{D}{r} + \frac{\mu'}{2} i^2 \tag{15}$$

per centimeter length of conductor A.

Putting the current i equal to unity and multiplying by 10^{-9} gives for the resultant inductance in henrys per conductor per centimeter length of line

$$L = \left(2 \log_e \frac{D}{r} + \frac{\mu'}{2}\right) 10^{-9} \text{ henrys} \tag{16}$$

In terms of common logarithms, the inductance per conductor per centimeter length of line is

$$L = \left(4.605 \log_{10} \frac{D}{r} + 0.5\mu'\right) 10^{-9} \text{ henrys} \tag{17}$$

For conductors of non-magnetic material such as are generally used for transmission lines, μ' is unity.

For a single-phase transmission line, with copper or aluminum conductors or with conductors of any other non-magnetic material, the reactance per conductor per 1000 feet is

$$x_{1000\ feet} = 2\pi f \left(140.4 \log_{10} \frac{D}{r} + 15.2\right) 10^{-6} \text{ ohms} \tag{18}$$

The reactance per mile per conductor is

$$x_{per\ mile} = 2\pi f \left(741 \log_{10} \frac{D}{r} + 80\right) 10^{-6} \text{ ohms} \tag{19}$$

It is important to observe that, for a single-phase transmission line, the reactance per conductor results from two parts, the self-inductance of the conductor considered and the mutual inductance due to the other conductor. For a polyphase line, the reactance per conductor also results from two factors, the self-inductance of the conductor considered and the mutual inductance due to the other conductors.

For a single-phase line, the flux due to the current i_A which links conductor B is equal to the flux due to the current i_B which links conductor A. Since the two currents, i_A and i_B, are not only equal but opposite in direction, the combined flux linkages produced by them with the current in each conductor are zero. The resultant inductance per conductor of a single-phase line, therefore, is due to the flux the current in the conductor considered would produce alone between its axis and the axis of the other conductor, assuming that the current is concentrated in the axes of the conductors.

Average Reactance per Conductor of a Completely Transposed, Ungrounded, Three-phase Transmission Line.—The lengths of the transposed sections are assumed equal. The average reactance per conductor per unit length of line is equal to the average flux linkages per ampere per conductor per unit length of line, due to both self-inductance and mutual inductance, multiplied by $2\pi f$, where f is the frequency of the line.

The transpositions required for complete transposition of the line are shown in Fig. 131. Parts (*a*), (*b*) and (*c*) of this figure show the three positions of the conductors. The D's with subscripts 1, 2 and 3 are the distances between the conductors, measured between centers. A, B and C are the conductors.

The inductance of any conductor, such as A, consists of three parts:

(a) That due to the flux linkages with A resulting from the flux produced by the current in conductor A, *i.e.*, by the current i_A. These linkages per unit current are its self-inductance (see page 400);

(b) That due to the flux linkages with A resulting from the flux produced by the current in conductor B, *i.e.*, by the current i_B. These linkages per unit current are the mutual inductance of B on A;

(c) That due to the flux linkages with A resulting from the flux produced by the current in conductor C, *i.e.*, by the current i_C. These linkages per unit current are the mutual inductance of C on A.

For the arrangement of the conductors shown in (a) of Fig. 131, the flux linkages with the current i_A in conductor A are, per centimeter length of line [see equations (9), (12) and (13)],

$$f.l._a = 2i_A^2 \log_\epsilon \frac{\infty}{r} + \frac{1}{2} i_A^2 + 2i_Ai_B \log_\epsilon \frac{\infty}{D_1} + 2i_Ai_C \log_\epsilon \frac{\infty}{D_3} \quad (20)$$

But

$$\log_\epsilon \frac{\infty}{r} \doteq \log_\epsilon \infty + \log_\epsilon \frac{1}{r}$$

$$\log_\epsilon \frac{\infty}{D_1} = \log_\epsilon \infty + \log_\epsilon \frac{1}{D_1}$$

$$\log_\epsilon \frac{\infty}{D_3} = \log_\epsilon \infty + \log_\epsilon \frac{1}{D_3}$$

Therefore,

$$f.l._a = 2i_A^2 \log_\epsilon \frac{1}{r} + \frac{1}{2} i_A^2 + 2i_Ai_B \log_\epsilon \frac{1}{D_1} + 2i_Ai_C \log_\epsilon \frac{1}{D_3} + 2i_A (i_A + i_B + i_C) \log_\epsilon \infty \quad (21)$$

Since $(i_A + i_B + i_C) = 0$, when there is no ground connection which carries current,

$$2i_A(i_A + i_B + i_C) \log_\epsilon \infty = 0 \quad (22)$$

The expression given in equation (22) is not indeterminate as might appear at first glance, but is actually zero. The linkages might equally well have been taken up to some distance such as x

from the conductor A, instead of up to infinity. If this had been done, the term in question would have been $2i_A\,(0)\log_\epsilon x$, which is obviously zero. Increasing x to infinity would not change the value of the expression, which would still be zero.

Since the last term in equation (21) is equal to zero,

$$f.l._a = 2i_A^2 \log_\epsilon \frac{1}{r} + \frac{1}{2}i_A^2 + 2i_Ai_B \log_\epsilon \frac{1}{D_1} + 2i_Ai_C \log_\epsilon \frac{1}{D_3} \quad (23)$$

For the arrangement of the conductors shown in (b) of Fig. 131, the flux linkages with current i_A in conductor A per centimeter length of line are

$$f.l._b = 2i_A^2 \log_\epsilon \frac{1}{r} + \frac{1}{2}i_A^2 + 2i_Ai_B \log_\epsilon \frac{1}{D_2} + 2i_Ai_C \log_\epsilon \frac{1}{D_1} \quad (24)$$

For the arrangement of the conductors shown in (c) of Fig. 131, the flux linkages with current i_A in conductor A per centimeter length of line are

$$f.l._c = 2i_A^2 \log_\epsilon \frac{1}{r} + \frac{1}{2}i_A^2 + 2i_Ai_B \log_\epsilon \frac{1}{D_3} + 2i_Ai_C \log_\epsilon \frac{1}{D_2} \quad (25)$$

The average flux linkages with current i_A in conductor A per centimeter length of line are

$$\begin{aligned} f.l._{av.} &= \frac{1}{3}\,(f.l._a + f.l._b + f.l._c) \\ &= 2i_A^2 \log_\epsilon \frac{1}{r} + \frac{1}{2}i_A^2 + \frac{2}{3}i_Ai_B \log_\epsilon \frac{1}{D_1 \times D_2 \times D_3} \\ &\qquad + \frac{2}{3}i_Ai_C \log_\epsilon \frac{1}{D_1 \times D_2 \times D_3} \end{aligned} \quad (26)$$

Since it is assumed that the line is ungrounded and has no neutral connection carrying current,

$$i_B + i_C = -i_A \quad (27)$$

and

$$\frac{2}{3}i_Ai_B \log_\epsilon \frac{1}{D_1 \times D_2 \times D_3} + \frac{2}{3}i_Ai_C \log_\epsilon \frac{1}{D_1 \times D_2 \times D_3} = \frac{2}{3}i_A^2 \log_\epsilon (D_1 \times D_2 \times D_3) \quad (28)$$

Combining equations (26) and (28) gives for the average flux linkages with the current i_A in conductor A per centimeter length of line

$$f.l._{av.} = i_A{}^2\left(\frac{1}{2} + 2\log_\epsilon\frac{1}{r} + \log_\epsilon\sqrt[3]{D_1{}^2 \times D_2{}^2 \times D_3{}^2}\right) \quad (29)$$

$$= i_A{}^2\left(\frac{1}{2} + \log_\epsilon\frac{\sqrt[3]{D_1{}^2 \times D_2{}^2 \times D_3{}^2}}{r^2}\right) \quad (30)$$

Similar expressions hold for conductors B and C.

It should be noticed that for a completely transposed three-phase line, the average flux linkages per conductor per unit length of line depend only on the current in the conductor considered. The resultant reactance drop per conductor per unit length of a completely transposed three-phase line depends only on the distances between the conductors and the current in the conductor considered. These statements assume that there is no neutral connection which carries current. If there is a neutral connection which carries current, equation (30) does not hold, since under these conditions equation (27) is not true.

The average reactance per conductor per centimeter length of a completely transposed three-phase line which has no ground or neutral connection carrying current is

$$x_{av.} = 2\pi f\left(\log_\epsilon\frac{\sqrt[3]{D_1{}^2 \times D_2{}^2 \times D_3{}^2}}{r^2} + \frac{1}{2}\right)10^{-9} \quad (31)$$

ohms per conductor per centimeter length of line.

Per mile of line in terms of common logarithms this becomes

$$x_{av.} = 2\pi f\left(370.6\log_{10}\frac{\sqrt[3]{D_1{}^2 \times D_2{}^2 \times D_3{}^2}}{r^2} + 80\right)10^{-6} \quad (32)$$

ohms per conductor per mile of line.

It is immaterial in what units the D's and r are expressed, provided they are expressed in the same unit.

If the conductors are at the corners of an equilateral triangle, $D_1 = D_2 = D_3 = D$ and

$$x_{av.} = 2\pi f\left(741\log_{10}\frac{D}{r} + 80\right)10^{-6} \quad (33)$$

ohms per conductor per mile of line.

By comparing equations (33) and (19), page 404, it is evident that the average reactance per conductor of a completely transposed three-phase line, with conductors at the corners of an equilateral triangle, is equal to the reactance per conductor of a

single-phase line of equal length and with equal spacing between conductors.

If the conductors of a completely transposed three-phase line are at the corners of an isosceles triangle, two of the distances D_1, D_2 and D_3 are equal. Two of the distances may also be equal with the conductors arranged in a plane. Let $D_1 = D_2 = D$ and let $D_3 = D'$. Then,

$$x_{av.} = 2\pi f\left\{370.6 \log_{10} \frac{\sqrt[3]{(D')^2 \times (D)^4}}{r^2} + 80\right\}10^{-6}$$

$$= 2\pi f\left\{370.6 \log_{10} \frac{\sqrt[3]{(D')^2 \times (D)^4}}{r^2} \times \frac{\sqrt[3]{(D)^2}}{\sqrt[3]{(D)^2}} + 80\right\}10^{-6}$$

$$= 2\pi f\left\{370.6 \left(2 \log_{10} \frac{D}{r} + \frac{2}{3} \log_{10} \frac{D'}{D}\right) + 80\right\}10^{-6}$$

$$= 2\pi f\left\{741 \log_{10} \frac{D}{r} + 80 + 247 \log_{10} \frac{D'}{D}\right\}10^{-6} \qquad (34)$$

ohms per conductor per mile of line.

When the conductors all lie in the same plane, with equal distances between the middle and each outside conductor, $D = \frac{D'}{2}$ and equation (34) becomes

$$x_{av.} = 2\pi f(741 \log_{10} \frac{D}{r} + 80 + 247 \log_{10} 2)\ 10^{-6}$$

$$= 2\pi f(741 \log_{10} \frac{D}{r} + 154)\ 10^{-6} \qquad (35)$$

ohms per conductor per mile of line.

When $D_1 = D_2 = D_3 = D$, *i.e.*, when the conductors are at the corners of an equilateral triangle, equations (23), (24) and (25), page 406, all reduce to

$$f.l. = 2i_A^2 \log_\epsilon \frac{1}{r} + \frac{1}{2}i_A^2 + 2i_A\ (i_B + i_C) \log_\epsilon \frac{1}{D} \qquad (36)$$

Since it is assumed that there is no neutral connection,

$$i_A + i_B + i_C = 0$$

$$-i_A = i_B + i_C \qquad (37)$$

Combining equations (36) and (37),

$$f.l. = 2i_A^2 \log_\epsilon \frac{D}{r} + \frac{1}{2}i_A^2 \qquad (38)$$

When the conductors are at the corners of an equilateral triangle, the flux linkages per conductor depend only on the current in the conductor considered and the distance D between conductors. The flux linkages per conductor and, therefore, the reactance per conductor of a three-phase line, with conductors at the corners of an equilateral triangle, are independent of the currents in the conductors other than the one considered. This is true whether the load carried by the line is balanced or unbalanced, provided there is no neutral connection which carries current. Transposition is not necessary to maintain balanced line drops with balanced load. Transposition is necessary, however, to prevent mutual inductance with other transmission lines or adjacent telephone lines.

Problem Illustrating the Calculation of the Reactance per Conductor of a Completely Transposed Three-phase Transmission Line.—A certain 110,000-volt, three-phase, 60-cycle transmission line has its conductors arranged in a horizontal plane with 10 feet between the middle conductor and each outside conductor. The conductors have a diameter of 0.46 inch. What is the reactance per conductor per mile of line?

From equation (35), page 408,

$$\begin{aligned} x_{av.} &= 2 \times 3.142 \times 60\left(741 \log_{10} \frac{10 \times 12}{0.23} + 154\right) 10^{-6} \\ &= 377\ (741 \times 2.72 + 154)\ 10^{-6} \\ &= 0.82 \text{ ohm per conductor per mile of line} \end{aligned}$$

Transfer of Power among the Conductors of a Three-phase Transmission Line.—There is a transfer of power among the conductors of a three-phase transmission line, except when the conductors are at the corners of an equilateral triangle. For simplicity, consider a three-phase line with conductors at the corners of an isosceles triangle. Refer to equations (23), (24) and (25), page 406. Let $D_1 = D_2 = D$ and let $D_3 = D'$.

Under these conditions, if the equations are divided by i_A and the remaining instantaneous values of the currents then are replaced by their vector values, the resulting equations multiplied by $2\pi f$ give the inductive voltage drops in conductor A in the three positions shown in Fig. 131. These drops are:

$$\bar{E}_a = 2\pi f\left\{2\bar{I}_A \log_\epsilon \frac{1}{r} + \frac{1}{2}\bar{I}_A + 2\left(\bar{I}_B \log_\epsilon \frac{1}{D} + \bar{I}_C \log_\epsilon \frac{1}{D'}\right)\right\}$$

$$= 2\pi f\left\{2\bar{I}_A \log_\epsilon \frac{D}{r} + \frac{1}{2}\bar{I}_A + 2\bar{I}_C \log_\epsilon \frac{D}{D'}\right\} \qquad (39)$$

$$\bar{E}_b = 2\pi f\left\{2\bar{I}_A \log_\epsilon \frac{1}{r} + \frac{1}{2}\bar{I}_A + 2(\bar{I}_B + \bar{I}_C) \log_\epsilon \frac{1}{D}\right\}$$

$$= 2\pi f\left\{2\bar{I}_A \log_\epsilon \frac{D}{r} + \frac{1}{2}\bar{I}_A\right\} \qquad (40)$$

$$\bar{E}_c = 2\pi f\left\{2\bar{I}_A \log_\epsilon \frac{1}{r} + \frac{1}{2}\bar{I}_A + 2\left(\bar{I}_B \log_\epsilon \frac{1}{D'} + \bar{I}_C \log_\epsilon \frac{1}{D}\right)\right\}$$

$$= 2\pi f\left\{2\bar{I}_A \log_\epsilon \frac{D}{r} + \frac{1}{2}\bar{I}_A + 2\bar{I}_B \log_\epsilon \frac{D}{D'}\right\} \qquad (41)$$

Let the vectors $\bar{I}_A$, $\bar{I}_B$ and $\bar{I}_C$, Fig. 132, represent the currents in the conductors A, B and C, respectively. Assume clockwise rotation and balanced conditions for the currents. Refer to equations (39) and (41).

Let D' be greater than D. Under this condition, $\log_\epsilon \frac{D}{D'}$ is negative. For the relative positions of the conductors shown in Fig. 131(*a*), page 403, there is a reactance drop $\bar{I}_A x = 2\pi f \bar{I}_A \left(2 \log_\epsilon \frac{D}{r} + \frac{1}{2}\right)$ [see equation (39)] in conductor A due to the current $\bar{I}_A$. This reactance drop is 90 degrees ahead of the current $\bar{I}_A$ and therefore represents no power with respect to $\bar{I}_A$. Since the logarithm of the ratio $\frac{D}{D'}$ is negative for the assumed relative magnitudes of D and D', the drop $\bar{I}_C x' = 2\pi f \bar{I}_C\left(2 \log_\epsilon \frac{D}{D'}\right)$ [see equation (39)] in conductor A due to the current $\bar{I}_C$ in conductor C is 90 degrees behind $\bar{I}_C$ as shown. It has components in phase and in quadrature with the current $\bar{I}_A$, Fig. 132. The effect of the quadrature component is to increase the apparent reactance drop in conductor A. The effect of the in-phase component is equivalent to an apparent increase in the resistance drop in conductor A. This apparent increase in the resistance drop in conductor A does not represent a loss of power. It merely represents a transfer of power from conductor A to conductor C by mutual inductance.

For the relative positions of the conductors shown in Fig. 131 (*b*), page 403, there is no mutual inductance between conductors *B* and *A* or between conductors *C* and *A*. [See equation (40).] There is therefore no transfer of power between conductors *B* and *A* or between conductors *C* and *A*.

For the relative positions of the conductors shown in Fig. 131 (*c*), page 403, there is a reactance drop $\bar{I}_A x = 2\pi f \bar{I}_A\left(2 \log_\epsilon \frac{D}{r} + \frac{1}{2}\right)$ [see equation (41)] in conductor *A* due to the current $\bar{I}_A$. This reactance drop is 90 degrees ahead of the current $\bar{I}_A$ and therefore represents no power with respect to $\bar{I}_A$. Since the logarithm of the ratio $\frac{D}{D'}$ is negative for the assumed relative magnitudes of *D* and *D'*, the drop $\bar{I}_B x' = 2\pi f \bar{I}_B\left(2 \log_\epsilon \frac{D}{D'}\right)$ [see equation (41)] in conductor *A* due to the current $\bar{I}_B$ in conductor *B* is 90 degrees behind $\bar{I}_B$ as shown. It has components both opposite in phase to, and in quadrature with, the current $\bar{I}_A$, Fig. 132. The effect of the quadrature component is to increase the apparent reactance drop in conductor *A*. The effect of the component which is opposite in phase to the current $\bar{I}_A$ is to produce an apparent decrease in the resistance drop in conductor *A*. This apparent decrease in resistance drop in the conductor *A* represents a transfer of power from conductor *B* to conductor *A* by mutual inductance.

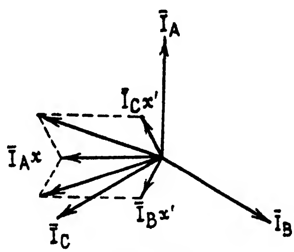


FIG. 132.

In a properly transposed line, the resultant transfer of power among conductors is zero for the line as a whole.

Mutual Inductance between Transmission Lines or between a Transmission Line and a Telephone Line.—The mutual inductance between a transmission line and a telephone line, or another transmission line which is parallel to the first line, may be found in exactly the same way as the mutual inductance between the conductors of a single transmission line.

Consider the mutual inductance between the conductors of a three-phase transmission line and one conductor of a telephone

line which is parallel to it. Let the currents in the transmission line be i_A, i_B and i_C and let i be the current in one conductor of the telephone line. Let D_A, D_B and D_C be the distances in centimeters between the conductor of the telephone line and the conductors of the transmission line which carry the currents i_A, i_B and i_C, respectively. Refer to Fig. 133.

The total flux linkages produced by the currents i_A, i_B and i_C with the current i in one conductor of the telephone line per centimeter length of the telephone line are, from equation (13), page 402,

$$
\begin{aligned}
f.l. &= 2i_A i \log_\epsilon \frac{\infty}{D_A} + 2i_B i \log_\epsilon \frac{\infty}{D_B} + 2i_C i \log_\epsilon \frac{\infty}{D_C} \\
&= 2i\left\{(i_A + i_B + i_C) \log_\epsilon \infty \right. \\
&\qquad \left. + i_A \log_\epsilon \frac{1}{D_A} + i_B \log_\epsilon \frac{1}{D_B} + i_C \log_\epsilon \frac{1}{D_C}\right\} \qquad (42)
\end{aligned}
$$

But $(i_A + i_B + i_C) = 0$, since there is no neutral connection which carries current. Therefore,

$$
f.l. = 2i\left\{i_A \log_\epsilon \frac{1}{D_A} + i_B \log_\epsilon \frac{1}{D_B} + i_C \log_\epsilon \frac{1}{D_C}\right\} \qquad (43)
$$

The voltage induced in abvolts per centimeter length of the conductor of the telephone line is

$$
\bar{V}_{abvolts} = 4\pi f\left\{\bar{I}_A \log_\epsilon \frac{1}{D_A} + \bar{I}_B \log_\epsilon \frac{1}{D_B} + \bar{I}_C \log_\epsilon \frac{1}{D_C}\right\} \qquad (44)
$$

where $\bar{I}_A$, $\bar{I}_B$ and $\bar{I}_C$ are the complex expressions for the currents in the conductors of the transmission line.

If the portion of the transmission line which is parallel to the telephone line is completely transposed, the resultant flux linkages with the conductor of the telephone line are zero, provided the transmission line has no neutral connection which carries current. If there is a neutral connection which carries current, the relation $(i_A + i_B + i_C) = 0$ does not hold.

With complete transposition, each conductor of the three-phase transmission line must occupy successively the positions 1, 2 and 3, Fig. 133. Each position must be maintained for one-third of the distance over which the complete transposition is

made. The conductors must be given a complete rotation in position.

The average flux linkages produced by the transposed transmission line are, per centimeter length of the conductor of the telephone line,

$$f.l._{av.} = \frac{2}{3}\left\{\left(i_A \log_\epsilon \frac{1}{D_A} + i_B \log_\epsilon \frac{1}{D_B} + i_C \log_\epsilon \frac{1}{D_C}\right) + \left(i_C \log_\epsilon \frac{1}{D_A} + i_A \log_\epsilon \frac{1}{D_B} + i_B \log_\epsilon \frac{1}{D_C}\right) + \left(i_B \log_\epsilon \frac{1}{D_A} + i_C \log_\epsilon \frac{1}{D_B} + i_A \log_\epsilon \frac{1}{D_C}\right)\right\} \quad (45)$$

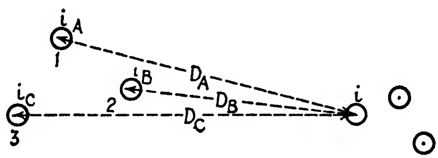


FIG. 133.

Since, for an ungrounded transmission line with no neutral connection, $(i_A + i_B + i_C) = 0$,

$$f.l._{av.} = \frac{2}{3}\left\{(i_A + i_B + i_C) \times \left(\log_\epsilon \frac{1}{D_A} + \log_\epsilon \frac{1}{D_B} + \log_\epsilon \frac{1}{D_C}\right)\right\} = 0 \quad (46)$$

Therefore, with complete transposition, the average mutual inductance between an ungrounded, three-phase transmission line and each conductor of a parallel telephone line is zero. The resultant voltage induced in each conductor of the telephone line therefore must be zero. The above statement is not limited to a telephone line but holds equally for the mutual inductance between a completely transposed three-phase transmission line with no neutral connection and any other parallel conductor. The parallel conductor may be one of the conductors of another transmission line.

The average mutual inductance between a three-phase power line and a telephone line may likewise be neutralized by transposing the conductors of the telephone line. If the conductors

of the telephone line are transposed, and the transmission line is not transposed, the voltages induced in adjacent transposed sections of the telephone line, by the mutual inductance of the transmission line, are opposite in phase. If the two lines are parallel and the transposed sections of the telephone line are equal in length, these voltages are equal in magnitude and in phase opposition, and their resultant therefore is zero. This is true without regard to the relative magnitudes of the currents, i_A, i_B and i_C, carried by the transmission line. Their sum may or may not be zero. There may or may not be a neutral connection which carries current. Therefore, transposing the telephone line eliminates the effect of mutual inductance, due to a three-phase transmission line, even when the transmission line has a ground connection which carries current.

Transposing the telephone line gives each of its conductors equal exposure to each of the conductors of the transmission line as well as to any neutral connection which may exist and must, therefore, make the resultant voltage induced in each conductor of the telephone line zero over any completely transposed length of telephone line.

Although it is possible to neutralize the mutual inductance between a transmission line and a telephone line by proper transposition of the lines, in practice complete neutralization of the mutual inductance is often difficult. In many cases the telephone line is not exactly parallel to the transmission line. Under such conditions, the transposed sections of the lines are at different distances from each other, and complete neutralization of the mutual inductance cannot be obtained unless the lengths of the transposed sections are varied or are made very short, neither of which is practicable. Moreover, in certain cases taps may be taken from the transmission line at certain points, making the currents unequal in the different transposed sections.

Voltage Induced in a Telephone Circuit by a Three-phase Transmission Line Which Is Parallel to the Telephone Line.—It is not necessary to find the linkages with each conductor of the telephone line. All that is required is the flux linkages with the loop formed by the two conductors of the telephone circuit.

Refer to Fig. 133, page 413. Let the currents which are there marked i_A, i_B and i_C be read as $\bar{I}_A$, $\bar{I}_B$ and $\bar{I}_C$, respectively, *i.e.*,

let them be vectors. In the figure, the distances from the conductors 1, 2 and 3 of the three-phase transmission line to one conductor i of the telephone line are D_A, D_B and D_C. Let one of the unlettered circles be the other conductor of the telephone line. Let the distances between this latter conductor and the three conductors of the three-phase line be D_A', D_B' and D_C'.

The total linkages with the telephone circuit per centimeter length of telephone line, due to the currents $\bar{I}_A$, $\bar{I}_B$ and $\bar{I}_C$ in the conductors of the transmission line, are

$$\bar{F}.\bar{L}.\text{ (circuit)} = 2\bar{I}_A \log_\epsilon \frac{D_A'}{D_A} + 2\bar{I}_B \log_\epsilon \frac{D_B'}{D_B} + 2\bar{I}_C \log_\epsilon \frac{D_C'}{D_C} \qquad (47)$$

The voltage induced by these flux linkages per unit length of telephone circuit is

$$\begin{aligned}\bar{V} &= 2\pi f(\bar{F}.\bar{L}.)\\ &= 2\pi f\left\{\bar{I}_A \log_\epsilon \frac{(D_A')^2}{(D_A)^2} + \bar{I}_B \log_\epsilon \frac{(D_B')^2}{(D_B)^2} + \bar{I}_C \log_\epsilon \frac{(D_C')^2}{(D_C)^2}\right\} \qquad (48)\end{aligned}$$

If the currents in equation (48) are expressed in abamperes, the voltage is in abvolts. If the currents are expressed in amperes and equation (48) is multiplied by 10^{-9}, the voltage is in volts.

The voltage induced per mile of telephone circuit, using common logarithms, is

$$\begin{aligned}\bar{V} = 2.33f\Big\{\bar{I}_A \log_{10} \frac{(D_A')^2}{(D_A)^2} &+ \bar{I}_B \log_{10} \frac{(D_B')^2}{(D_B)^2}\\ &+ \bar{I}_C \log_{10} \frac{(D_C')^2}{(D_C)^2}\Big\}\ 10^{-3} \text{ volts} \qquad (49)\end{aligned}$$

The currents in equation (49) must be expressed in amperes and in their complex form.

It is convenient in equations (48) and (49) to put the 2's, which are coefficients of the logarithms in equation (47), as exponents of the D's. This simplifies the use of the equations. In most cases in practice, the distances between the conductors of adjacent transmission and telephone lines are not directly known, but they can easily be found from the known horizontal and vertical spacing of the conductors and the height of poles and distance between poles of the two lines. Putting the 2's in the equations as exponents of the D's, instead of as coefficients

of the logarithms, is convenient, as it avoids taking the square roots which would be necessary in finding the distances between the conductors. The squares of the distances between the conductors are readily found by taking the sum of the squares of the horizontal and vertical distances between them.

It must not be forgotten that equations (47), (48) and (49) are vector equations and that all terms in them must be added vectorially. The currents should be expressed in their complex form, referred to some conveniently chosen axis.

Voltage Induced in a Telephone Circuit by an Adjacent Three-phase Transmission Line Which Carries an Unbalanced Load.—Equation (49) holds for either balanced or unbalanced loads. However, when a careful study or analysis of the inductive effects of a transmission line on a telephone line is to be made, it is best to resolve the unbalanced load currents carried by the transmission line into positive-phase, negative-phase and zero-phase components. (See Chapter XIII.) The zero-phase components, when they exist, cause by far the most trouble, since they are all in phase. They are usually called the *residuals* by telephone engineers. Since the residuals are all in phase, no amount of transposition of the transmission line diminishes their inductive effect on an adjacent telephone line. To get rid of the inductive effect of the residuals, the telephone line itself must be transposed. Since the vector sum of the positive-phase components and also the vector sum of the negative-phase components are each zero, they would produce no inductive effect on an adjacent telephone line if the conductors of the transmission line could be equidistant from each conductor of the telephone line. This statement would not apply to the residuals, since they are in phase.

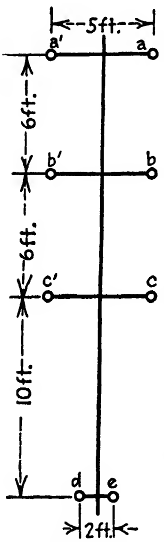


FIG. 134.

Example of the Calculation of the Voltage Induced in a Telephone Line by Electromagnetic Induction.—Two three-phase, 60-cycle transmission lines, a, b, c and a', b', c', and a telephone line d, e are carried on the same poles with spacings between conductors shown in Fig. 134. The conductors of the transmis-

sion line are No. 000 B. & S. gauge wire and have a diameter of 0.41 inch. The transmission lines are paralleled at the generating station, with corresponding conductors connected to the same phase. The phase order is a, b, c, with the current in phase a leading the current in phase b. Find the voltage induced electromagnetically in the telephone line, per mile, when each of the three-phase transmission lines carries a balanced load with 200 amperes per conductor. The power factors of lines a, b, c and a', b', c' are unity and 0.85 (lagging current), respectively. Also, calculate the voltage induced in the telephone line when each transmission line is operated with an inductive load of 0.80 power factor and 200 amperes per conductor.

The squares of the distances between the conductors of the transmission lines and the conductors of the telephone line are given below:

$$
\begin{aligned}
(ad)^2 &= (a'e)^2 = (22)^2 + (3.5)^2 = 496.3 \text{ feet squared}\\
(ae)^2 &= (a'd)^2 = (22)^2 + (1.5)^2 = 486.3 \text{ feet squared}\\
(bd)^2 &= (b'e)^2 = (16)^2 + (3.5)^2 = 268.3 \text{ feet squared}\\
(be)^2 &= (b'd)^2 = (16)^2 + (1.5)^2 = 258.3 \text{ feet squared}\\
(cd)^2 &= (c'e)^2 = (10)^2 + (3.5)^2 = 112.3 \text{ feet squared}\\
(ce)^2 &= (c'd)^2 = (10)^2 + (1.5)^2 = 102.3 \text{ feet squared}
\end{aligned}
$$

Let φ with subscripts a, b, c and a', b', c' be the flux linking the telephone line, per centimeter length of telephone line, due to the currents in the conductors a, b, c and a', b', c', respectively. The magnitudes of these fluxes are:

$$
\begin{aligned}
\varphi_a &= \varphi_{a'} = \frac{200}{10}\log_\epsilon\left(\frac{ad}{ae}\right)^2 = 20 \times 2.303 \times 0.00884 = 0.407\\
\varphi_b &= \varphi_{b'} = \frac{200}{10}\log_\epsilon\left(\frac{bd}{be}\right)^2 = 20 \times 2.303 \times 0.0165 = 0.760\\
\varphi_c &= \varphi_{c'} = \frac{200}{10}\log_\epsilon\left(\frac{cd}{ce}\right)^2 = 20 \times 2.303 \times 0.0405 = 1.866
\end{aligned}
$$

The resultant flux through the telephone circuit per unit length of line is equal to the vector sum of the component fluxes φ_a, φ_b and φ_c, minus (vectorially) the vector sum of the component fluxes $\varphi_{a'}$, $\varphi_{b'}$ and $\varphi_{c'}$.

$$
\bar{\varphi}_0 = \bar{\varphi}_a - \bar{\varphi}_{a'} + \bar{\varphi}_b - \bar{\varphi}_{b'} + \bar{\varphi}_c - \bar{\varphi}_{c'} \qquad (50)
$$

$\bar{\varphi}_{a'}$ lags $\bar{\varphi}_a$, $\bar{\varphi}_{b'}$ lags $\bar{\varphi}_b$ and $\bar{\varphi}_{c'}$ lags $\bar{\varphi}_c$ by $\cos^{-1} 0.85 = 31.8$ degrees. $\bar{\varphi}_a$ leads $\bar{\varphi}_b$ by 120 degrees and leads $\bar{\varphi}_c$ by 240 degrees. $\bar{\varphi}_{a'}$ leads $\bar{\varphi}_{b'}$ by 120 degrees and leads $\bar{\varphi}_{c'}$ by 240 degrees.

Use the flux $\bar{\varphi}_a$ as an axis of reference. Then the resultant flux linking the telephone circuit, per centimeter length of the telephone circuit, is

$$\begin{aligned}
\bar{\varphi}_0 &= \varphi_a(\cos 0° - j \sin 0°) - \varphi_{a'}(\cos 31.8° - j \sin 31.8°) \\
&\quad + \varphi_b(\cos 120° - j \sin 120°) - \varphi_{b'}(\cos 151.8° - j \sin 151.8°) \\
&\quad + \varphi_c(\cos 240° - j \sin 240°) - \varphi_{c'}(\cos 271.8° - j \sin 271.8°) \\
&= 0.407 - j\,0.000 - 0.346 + j\,0.214 - 0.380 - j\,0.658 \\
&\quad + 0.670 + j\,0.359 - 0.934 + j\,1.616 - 0.059 - j\,1.865 \\
&= -0.642 - j\,0.334 \\
\varphi_0 &= \sqrt{(0.642)^2 + (0.334)^2} \\
&= 0.724 \text{ maxwell}
\end{aligned}$$

The flux linkages per centimeter length of telephone line are equal to the flux per centimeter length of line, since the telephone line is a circuit of one turn.

The voltage induced in the telephone circuit per mile of telephone line is

$$\begin{aligned}
V_{per\ mile} &= 2\pi f(0.724)\,10^{-8} \times 2.54 \times 12 \times 5280 \\
&= 0.439 \text{ volt per mile}
\end{aligned}$$

It is obvious from equation (50) that if the loads carried by the transmission lines are both balanced and have the same power factor, the resultant flux linkages with the telephone line are zero. Under this condition, the voltage induced in the telephone line is also zero. Therefore, when both transmission lines operate with balanced inductive loads of 0.80 power factor and 200 amperes per line, the voltage induced in the telephone line is zero.

Third-harmonic Voltage Induced in a Telephone Line Adjacent to a Transmission Line Fed from a Bank of Transformers Connected in Wye on Both Primary and Secondary Sides.—The exciting current in any transformer with an iron core is not sinusoidal, even when the impressed voltage is sinusoidal. Owing to the variation in the permeability of the core during a cycle, the exciting current always contains harmonics, among which the

third is very prominent. This third-harmonic component of the exciting current may be as large as 40 or 50 per cent of the fundamental of the no-load current.

The third harmonics in a balanced three-phase system are all in phase. (See Chapter XI.) If the primary and secondary windings of a bank of transformers are both connected in wye, with the neutral on the primary sides connected to the source of power, the necessary third harmonics in the exciting currents of the transformers have a common return path to the source of power over the neutral connection. Since these harmonics are in phase, they add directly on the neutral. They cannot exist if the neutral connection is opened.

When no neutral connection is used on the primary side, there can be no third harmonic in the exciting currents. There are consequently marked third harmonics in the fluxes of the transformers and in the voltages induced in the primary and secondary windings. These third-harmonic voltages cancel between the line terminals. (See Chapter XI.) They exist in the phase voltages between the line terminals and neutral.

If the neutral on the secondary side of the transformers is grounded, the third harmonics in the phase voltages, *i.e.*, the voltages to neutral, cause a third-harmonic charging current in each conductor of the transmission line due to its capacitance to ground. These charging currents are in phase. Consequently, their inductive effects on an adjacent telephone line are directly additive. The fundamental and all harmonics in the transmission line, except the third harmonic and its multiples, differ by 120 degrees in time phase. Their inductive effects on any adjacent telephone line therefore tend to neutralize. Their resultant inductive effect would be zero if the conductors of the adjacent telephone line could be equidistant from each of the conductors of the transmission line.

Due to the fact that the inductive effects produced by the third harmonics add directly, and also because of their triple frequency, the voltage induced by them in an adjacent telephone line is much greater than the voltage induced by fundamental currents of like magnitude. A given amount of third-harmonic voltage in a telephone line produces much more interference with the transmission of speech than an equal amount of funda-

mental voltage, since the third-harmonic frequency approaches that of the voice waves.

What has been said regarding the third harmonic applies to any multiple of the third harmonic, *i.e.*, the ninth, fifteenth etc. The multiples of the third harmonic in the exciting currents of transformers, however, are as a rule small compared with the third harmonic itself. The presence of ninth harmonics, or more especially fifteenth harmonics, would cause very serious interference with telephonic transmission of speech, since their frequencies are close to the resonant frequency of the telephone receiver, which is about 870 cycles per second.

CHAPTER XV

CAPACITANCE OF A TRANSMISSION LINE

Capacitance of Two Straight Parallel Conductors.—Consider two straight, parallel, metallic filaments A and B at a distance d centimeters apart in a uniform medium of dielectric constant K. Assume that the filaments are of zero radius. Let the charges per centimeter length of A and B be, respectively, $+q$ and $-q$ electrostatic units of electricity. For such a system of charges, the distribution of field intensity and the distribution of potential are the same in all planes perpendicular to the axes of the filaments. Hence, in treating the electrostatic problems which arise from such a distribution, it is sufficient to consider the relations existing in one such plane. This is the method which will be followed in this chapter.

The field intensity $\mathcal{H}$ at a point distant r centimeters from a straight filament of infinite length charged with $+q$ electrostatic units of electricity per centimeter length is

$$\mathcal{H} = \frac{1}{K} \times \frac{2q}{r} = -\frac{dv}{dr} \text{ dynes per unit charge} \tag{1}$$

The algebraic sign of the field intensity is determined by the direction in which a free positive charge would move if placed in the field. A free positive charge tends to move in the direction of diminishing potential. The field intensity $\mathcal{H}$, therefore, is positive in a direction away from the charge $+q$. This is indicated by the minus sign before $\frac{dv}{dr}$, which is negative, since v is decreasing in the direction of increasing r.

It should be remembered in what follows that the potential difference between two points is measured by the work done in moving a unit positive charge from one point to the other. Potential, therefore, in its fundamental conception cannot be a space vector. If its magnitude varies periodically with time, as

in alternating-current circuits, it may then be considered as a time vector.

Refer to Fig. 135. Let P be any point distant r_A and r_B from the filaments A and B, respectively. Let R be a point equidistant from the filaments and at a distance a centimeters from each. The difference in potential between the points R and P, due to the charged filament A, is

$$v_{PR} = \frac{1}{K}\int_{r_A}^{a} \frac{2q}{r}\,dr$$

$$= \frac{2q}{K}\log_\epsilon \frac{a}{r_A} \tag{2}$$

where v_{PR} is the potential difference between the points R and P, due to the charge $+q$ per unit length of A, and is measured by

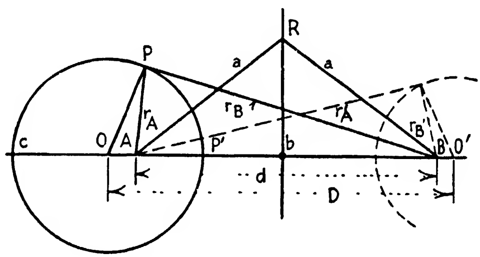


FIG. 135.

the work done in carrying a unit positive charge from R to P. If v_{PR}' is the potential difference between these points, due to the charge $-q$ per unit length of B, then

$$v_{PR}' = \frac{1}{K}\int_{r_B}^{a} \frac{-2q}{r}\,dr$$

$$= -\frac{2q}{K}\log_\epsilon \frac{a}{r_B}$$

$$= \frac{2q}{K}\log_\epsilon \frac{r_B}{a} \tag{3}$$

Since the resultant potential difference between any two points, due to any number of charges, is equal to the algebraic sum of the potential differences due to each charge separately, the difference in potential between the points R and P must be

$$v_{PR} + v_{PR}' = v$$
$$= \frac{2q}{K}\log_e \frac{r_B}{r_A} \tag{4}$$

If the ratio $\frac{r_A}{r_B} = k$ is constant, equation (4) is the equation of an equipotential line and is a circle. In any plane perpendicular to the axes of the filaments, the equipotential lines are circles. The actual equipotential surfaces due to the filaments are, of course, circular cylinders. That the equipotential lines are circles may be shown as follows. Refer the point P, Fig. 135, to rectangular coördinates, taking the axes of X and Y along, and at right angles to, the line joining A and B. Let the origin be midway between the points A and B, at b. Then,

$$r_A^2 = \left(\frac{d}{2} + x\right)^2 + y^2$$
$$r_B^2 = \left(\frac{d}{2} - x\right)^2 + y^2$$
$$\frac{r_A^2}{r_B^2} = \frac{\left(\frac{d}{2} + x\right)^2 + y^2}{\left(\frac{d}{2} - x\right)^2 + y^2} = k^2 = \text{constant} \tag{5}$$
$$\left(\frac{d}{2}\right)^2 + 2\frac{d}{2}x + x^2 + y^2 = k^2\left\{\left(\frac{d}{2}\right)^2 - 2\frac{d}{2}x + x^2 + y^2\right\}$$
$$\left(\frac{d}{2}\right)^2(1 - k^2) + 2\frac{d}{2}x\,(1 + k^2) + x^2(1 - k^2) + y^2(1 - k^2) = 0$$
$$\left(\frac{d}{2}\right)^2 + 2\frac{d}{2}x\left(\frac{1 + k^2}{1 - k^2}\right) + x^2 + y^2 = 0 \tag{6}$$

Equation (6) is the equation of a circle. Completing the square gives

$$x^2 + 2\frac{d}{2}x\left(\frac{1+k^2}{1-k^2}\right) + \left(\frac{d}{2}\right)^2\left(\frac{1+k^2}{1-k^2}\right)^2 + y^2 = \left(\frac{d}{2}\right)^2\left\{\left(\frac{1+k^2}{1-k^2}\right)^2 - 1\right\}$$
$$\left\{x + \frac{d}{2}\left(\frac{1 + k^2}{1 - k^2}\right)\right\}^2 + y^2 = \left(\frac{d}{2}\right)^2\left\{\left(\frac{1 + k^2}{1 - k^2}\right)^2 - 1\right\} \tag{7}$$
$$OP = OP' = \frac{d}{2}\sqrt{\left(\frac{1 + k^2}{1 - k^2}\right)^2 - 1} = \frac{d}{1 - k^2}k \tag{8}$$

is the radius of the circle.

$$bO = -\frac{d}{2}\left(\frac{1 + k^2}{1 - k^2}\right) \tag{9}$$

is the distance of the center of the circle from the origin b. It is to be noted that the center of the circle lies on the axis of X (Fig. 135). The center lies at a distance

$$bO - bA = -\frac{d}{2}\left(\frac{1 + k^2}{1 - k^2}\right) + \frac{d}{2}$$

$$= -d\left(\frac{k^2}{1 - k^2}\right) \tag{10}$$

from A, or at a distance

$$BA + AO = -d - d\left(\frac{k^2}{1 - k^2}\right)$$

$$= -d\left(\frac{1}{1 - k^2}\right) \tag{11}$$

from B.

The equipotential lines in any plane perpendicular to the axes of the filaments are two groups of circles, one surrounding A, the other surrounding B, with their centers at a distance $d\left(\frac{k^2}{1 - k^2}\right)$ to the left of A when $\frac{r_A}{r_B} = k$ is less than 1, and at a distance $d\left(\frac{k^2}{1 - k^2}\right)$ to the right of B when $\frac{r_A}{r_B} = k$ is greater than 1. Hence, the equipotential lines in any plane perpendicular to the axes of the filaments are circles whose centers lie on the axis of X, *i.e.*, on a line through the axes of the filaments. The actual equipotential surfaces due to the filaments are circular cylinders whose axes are parallel to the axes of the filaments.

When $\frac{r_A}{r_B} = 1$, the radius of the equipotential line becomes infinite [equation (8)], and the line is a straight line passing through b perpendicular to the line joining the axes of the filaments. The corresponding equipotential surface is an infinite plane passing through b and perpendicular to the line joining the axes of the filaments.

Since the surface of any conductor must be equipotential, a conducting cylinder may be introduced in the field due to the charges on A and B with its surface coincident with any equipotential surface, without altering the field. If the equipotential surface in question is one of the group which surrounds A, the charge on the filament A may be considered as existing on the

surface of such a cylinder. As the surface of any conducting body must be equipotential, the charge distributes itself in such a way as to make the surface of the cylinder equipotential. The effect at any point outside of the cylinder is the same as if the charge were concentrated along the filament A. It follows that any two parallel, straight conductors of circular cross section, having equal and opposite charges per unit length, can be replaced by two charged filaments at the points A and B, Fig. 135, page 422, provided the charges are not influenced by the presence of any other charges. The positions of the points A and B are determined by the radii of the cylinders and their distance apart in accordance with the expressions given in equations (8) to (11), inclusive.

The equipotential surfaces due to two parallel, straight, charged, conducting, circular cylinders are shown in Fig. 136.

The solid lines in this figure are the intersections of the equipotential surfaces with a plane perpendicular to the two charged cylinders. The solid heavy circles are the projections of the cylinders on this plane. The dots in the circles represent the projections of the filaments. The lines of electrostatic force due to the charges on the cylinders are shown dotted. These lines must be everywhere perpendicular to the corresponding equipotential surfaces. It can be shown that the lines of force are also circles.

Remembering that the two cylinders have equal and opposite charges per unit length, it follows from equation (4), page 423, that the difference in potential between B and A is

$$
\begin{aligned}
v &= \frac{2q}{K}\left(\log_\epsilon \frac{r_B}{r_A} - \log_\epsilon \frac{r_B'}{r_A'}\right) \\
&= \frac{2q}{K} \log_\epsilon \frac{r_B r_A'}{r_A r_B'} \qquad (12)
\end{aligned}
$$

where r_A' and r_B' are the distances for the cylinder B corresponding to the distances r_A and r_B for the cylinder A. (See Fig. 135, page 422.)

Since the capacitance of any two conducting bodies, which are removed from the influence of all other charged bodies, is equal to the ratio of the charge on either to the difference of potential between them, the capacitance of two parallel, straight cylinders

or conductors of circular cross section, which are removed from all other charged bodies, must be

$$C = \frac{q}{v} = \frac{q}{\dfrac{2q}{K}\log_\epsilon \dfrac{r_B r_A'}{r_A r_B'}}$$

$$= \frac{K}{2\log_\epsilon \dfrac{r_B r_A'}{r_A r_B'}} \tag{13}$$

C is the capacitance in electrostatic units per centimeter length of the conducting cylinders. Equation (13) is not in a convenient

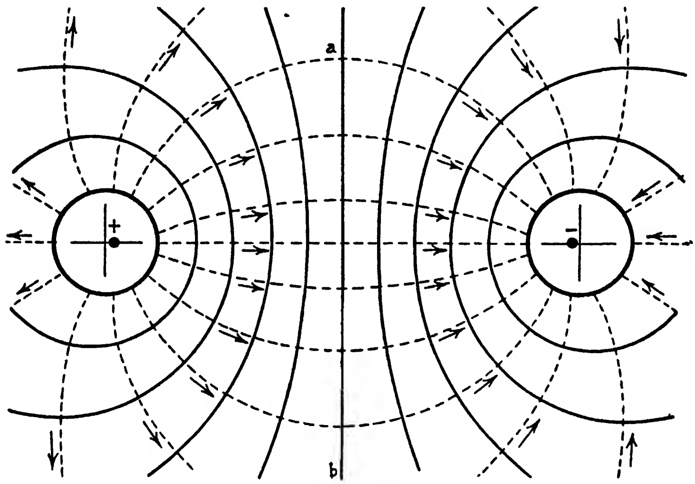


FIG. 136.

form for use, since r_A, r_B, r_A' and r_B' are all variables. These may be replaced in terms of the radii r_a and r_b of the cylinders and the distance between their centers.

The radii of the cylinders are given by equation (8), page 423:

$$\text{radii} = \frac{d}{1 - k^2}k$$

This is positive when $\dfrac{r_A}{r_B} = k$ is less than unity. It is negative when $\dfrac{r_A}{r_B} = k$ is greater than unity. For the equipotential surface corresponding to the outer surface of the conductor surround-

ing A, $\frac{r_A}{r_B} = k$ is less than unity and the expression for its radius is positive. Therefore, if r_a is the radius of the conductor,

$$r_a = \frac{d}{1 - k^2} k$$

and

$$k = \frac{r_A}{r_B} = -\frac{d}{2r_a} + \sqrt{\left(\frac{d}{2r_a}\right)^2 + 1} \qquad (14)$$

For the equipotential surface corresponding to the outer surface of the conductor surrounding B, $\frac{r_A'}{r_B'} = k_1$ is greater than unity and the expression for its radius is negative. Therefore, if r_b is the radius of the conductor,

$$-r_b = \frac{d}{1 - k_1^2} k_1$$

and

$$k_1 = \frac{r_A'}{r_B'} = \frac{d}{2r_b} + \sqrt{\left(\frac{d}{2r_b}\right)^2 + 1} \qquad (15)$$

Substituting in equation (13) the values of $\frac{r_A}{r_B}$ and $\frac{r_A'}{r_B'}$ from equations (14) and (15), respectively, gives for the capacitance

$$C = \frac{K}{2\left\{\log_\epsilon\left[\frac{d}{2r_b} + \sqrt{\left(\frac{d}{2r_b}\right)^2 + 1}\right] - \log_\epsilon\left[-\frac{d}{2r_a} + \sqrt{\left(\frac{d}{2r_a}\right)^2 + 1}\right]\right\}} \qquad (16)$$

Equation (16) gives the capacitance in electrostatic units per centimeter length of line. Since d, r_a and r_b enter only in ratios, it is immaterial in what units they are expressed, provided they are all expressed in the same unit.

The distance d between the filaments may easily be found in terms of the radii r_a and r_b of the conductors and the distance D between their centers. Refer to Fig. 135, page 422. Since $\frac{r_A}{r_B}$ is constant and $P'Pc$ is a circle,

$$\frac{r_A}{r_B} = \frac{Ac}{cB} = \frac{AP'}{P'B}$$

$$\frac{Ac + AP'}{cB + P'B} = \frac{Ac - AP'}{cB - P'B}$$

$$\frac{2r_a}{2r_a + 2P'B} = \frac{2\,OA}{2r_a}$$

$$\frac{r_a}{r_a + d - (r_a - OA)} = \frac{OA}{r_a}$$

$$r_a^2 = d(OA) + (OA)^2$$

$$OA = -\frac{d}{2} + \sqrt{r_a^2 + \left(\frac{d}{2}\right)^2} \tag{17}$$

OA is the offset of the filament A from the center of the conductor about A.

The offset $O'B$ of the filament B from the center of the conductor about B may be found in a similar manner. It is

$$O'B = -\frac{d}{2} + \sqrt{r_b^2 + \left(\frac{d}{2}\right)^2} \tag{18}$$

Then,

$$D = d + OA + O'B$$
$$= \sqrt{r_a^2 + \left(\frac{d}{2}\right)^2} + \sqrt{r_b^2 + \left(\frac{d}{2}\right)^2} \tag{19}$$

Solving equation (19) for d gives

$$d^2 = \frac{D^4 + (r_a^2 - r_b^2)^2 - 2D^2(r_a^2 + r_b^2)}{D^2}$$
$$= D^2 + \frac{(r_a^2 - r_b^2)^2}{D^2} - 2(r_a^2 + r_b^2) \tag{20}$$

Equation (20) may be put in the following form:

$$d^2 = \frac{(D + r_a + r_b)(D + r_a - r_b)(D - r_a + r_b)(D - r_a - r_b)}{D^2} \tag{21}$$

By calculating the numerical value of d from either equation (20) or (21) and substituting this and the values of the radii in equation (16), the capacitance of two parallel, straight conductors of circular cross section may be found in electrostatic units per centimeter length of the conductors.

Equations (16) and (21) may be combined and reduced to the following form:

$$C = \frac{1}{2 \log_\epsilon (\alpha + \sqrt{\alpha^2 - 1})} \tag{22}$$
$$= \frac{1}{2 \cosh^{-1} \alpha} \tag{23}$$

where

$$\alpha = \frac{D^2 - r_a^2 - r_b^2}{2r_a r_b} \tag{24}$$

When the distance D between the conductors is great compared with their radii r_a and r_b, d is very nearly equal to D. [See equation (20).]

The conductors of a transmission line are of equal radius and are in air. Therefore, for a transmission line, $K = 1$ and $r_a = r_b = r$. When the radii of the conductors are equal, d, from equation (21), is

$$d = \sqrt{(D + 2r)(D - 2r)} = \sqrt{D^2 - 4r^2}$$

Substituting this value of d and $r_a = r_b = r$ in equation (16), page 427,

$$C = \frac{1}{2\left\{\log_\epsilon\left[\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1}\right] - \log_\epsilon\left[\frac{D}{2r} - \sqrt{\left(\frac{D}{2r}\right)^2 - 1}\right]\right\}}$$

$$= \frac{1}{2 \log_\epsilon \dfrac{\left\{\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1}\right\}}{\left\{\frac{D}{2r} - \sqrt{\left(\frac{D}{2r}\right)^2 - 1}\right\}}}$$

$$= \frac{1}{4 \log_\epsilon \left\{\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1}\right\}} \tag{25}$$

$$= \frac{1}{4 \cosh^{-1} \dfrac{D}{2r}} \tag{26}$$

Since, for transmission lines, D is large compared with r, $\sqrt{\left(\frac{D}{2r}\right)^2 - 1}$ is very nearly equal to $\left(\frac{D}{2r}\right)$ and may be assumed equal to $\left(\frac{D}{2r}\right)$, especially as $\sqrt{\left(\frac{D}{2r}\right)^2 - 1}$ enters in a logarithm. Putting

$$\sqrt{\left(\frac{D}{2r}\right)^2 - 1} = \frac{D}{2r}$$

in equation (25),

$$C = \frac{1}{4 \log_\epsilon \dfrac{D}{r}} \tag{27}$$

for the capacitance of a single-phase transmission line in electrostatic units per centimeter length of line.

It is convenient to reduce equation (27) to electromagnetic units, either per 1000 feet or per mile of line. It is also convenient to have the capacitance expressed in terms of common logarithms. From the dimensions of the electrostatic and electromagnetic units of capacitance, it may easily be shown that the ratio of the electromagnetic unit of capacitance to the electrostatic unit of capacitance is equal to the square of a velocity. This velocity has been shown to be the velocity of light, or 3×10^{10} centimeters per second. Therefore, to reduce a given capacitance expressed in electrostatic units to its equivalent capacitance in electromagnetic units, it is necessary to divide by the square of the velocity of light, or by $(3 \times 10^{10})^2$.

Hence, for a single-phase line consisting of two equal, parallel, straight conductors of circular cross section, to get the capacitance in microfarads per 1000 feet of line and in terms of common logarithms, multiply the capacitance given by equation (27) by

$$\frac{1}{(3 \times 10^{10})^2} \times (2.540 \times 12 \times 1000) \times \frac{1}{2.303} \times (10^9 \times 10^6)$$
$$= 14.706 \times 10^{-3}$$

$$C_{microfarads} = \frac{14.7 \times 10^{-3}}{4 \log_{10} \dfrac{D}{r}} = \frac{3.68 \times 10^{-3}}{\log_{10} \dfrac{D}{r}} \tag{28}$$

microfarads per 1000 feet of line. Per mile of line this becomes

$$C_{microfarads} = \frac{19.4 \times 10^{-3}}{\log_{10} \dfrac{D}{r}} \tag{29}$$

microfarads per mile of line.

Equation (29) would be exact if the surface densities of the charges on the cylinders were uniform. In this case, the positions of the filaments on which the charges may be considered concentrated would coincide with the axes of the cylinders. The

distribution of the charges on coaxial cylinders would be uniform.

In all cases occurring in practice, where it is necessary to determine the capacitance or the charging current for a transmission line, the distances between the conductors and between the conductors and the earth are so great compared with the diameters of the conductors that the distribution of the charges on the conductors may be assumed to be uninfluenced except by the shape of the conductors themselves. Since the conductors of transmission lines are nearly always circular in cross section, the charges on the conductors of transmission lines may be assumed to be concentrated on the axes of the conductors. Equation (29) may be applied in the case of all aerial transmission lines without sensible error.

Example.—A certain two-wire transmission line has the conductors spaced 10 feet on centers. The conductors have a radius of 0.285 inch. What is the capacitance of the line between conductors in microfarads per mile of line?

$$\begin{aligned} C &= \frac{19.4 \times 10^{-3}}{\log_{10} \dfrac{D}{r}} \\ &= \frac{19.4 \times 10^{-3}}{\log_{10} \dfrac{10 \times 12}{0.285}} \\ &= \frac{19.4 \times 10^{-3}}{2.6243} \\ &= 7.39 \text{ microfarads per mile} \end{aligned}$$

Capacitance between the Earth and a Straight Conductor Parallel to Its Surface.—The capacitance between the earth and a straight conductor parallel to its surface may be calculated on the assumption that the earth is an equipotential surface of zero potential. Since the earth is a conductor, the electrostatic lines of force must enter it perpendicular to its surface. It can be shown that the electrostatic field between an infinite conducting plane and a charged straight conductor which is parallel to this plane is the same as would be produced between one of two parallel, straight, charged conductors and the equipotential surface midway between them. The surface of the earth may be assumed to be the equipotential surface *ab*, Fig. 136, page

426. According to this assumption, the capacitance between the earth and a conductor parallel to it at a perpendicular distance h above its surface may be calculated by assuming a hypothetical conductor or *image* parallel to the actual conductor and situated at a distance h below the surface of the earth. This hypothetical conductor or image has a charge equal and opposite to the charge on the actual conductor. The charge which is actually on the surface of the earth is assumed to be on the hypothetical conductor or image.

Since the surface of the earth corresponds to the equipotential surface ab, Fig. 136, between the actual conductor A and its image B, the difference of potential between the earth and the actual conductor is one-half as great as that between the conductor and its image. Therefore, since $C = \frac{Q}{V}$, the capacitance of the conductor with respect to the earth is twice that given by equations (28) and (29), page 430, or

$$C_{to\ earth} = \frac{7.36 \times 10^{-3}}{\log_{10} \frac{2h}{r}} \text{ microfarads per 1000 feet} \qquad (30)$$

$$= \frac{38.8 \times 10^{-3}}{\log_{10} \frac{2h}{r}} \text{ microfarads per mile} \qquad (31)$$

When applying the formulas for capacitance to earth, it should be remembered that the earth in the neighborhood of the conductor is assumed to be a level plane of perfectly conducting material. The assumed distribution of the electric field, and therefore the calculation of the capacitance, may be considerably modified by the presence of buildings, trees and poles, especially steel poles. It is also influenced by the variation in the height of the conductor above the ground, as well as by the fact that the earth, in certain localities or in dry weather, differs considerably from a perfect conductor. It has been shown experimentally that the equipotential plane corresponding to the surface of the earth actually lies below the earth's surface. Experiments carried out at the Massachusetts Institute of Technology in Cambridge have shown it there to be approximately coincident with the mean tide-water level.

Charging Current of a Transmission Line.—Consider a three-phase transmission line with all three conductors in a single plane which is parallel to the surface of the earth. Assume the surface of the earth to be a perfectly conducting plane. According to this assumption, the image of any conductor, as for example conductor 1, Fig. 137, is at a distance below the surface of the earth equal to the height h of the conductor above the surface. A line joining any conductor and its image must be perpendicular to the surface of the earth, since this surface forms the equipotential plane of zero potential. (See Fig. 136, page 426.)

Let the distances between the middle conductor and each of the outside conductors be equal. Call this distance D. Let h be the height of the conductors above the earth. Assume the distances between the conductors and also the height of the conductors above the earth to be large compared with their radii. Under these conditions, the charges on the conductors may be considered as concentrated along the axes of the conductors, so far as any effect outside of the conductors is concerned. Figure 137 shows the conductors of the line and their images.

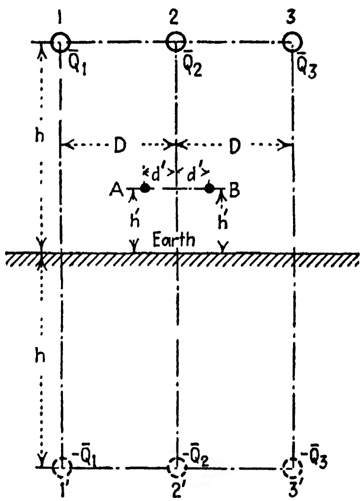


FIG. 137.

All charges and voltages are expressed as vectors. Let the charges on the conductors per centimeter length of line be $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ electrostatic units. The corresponding charges on the images are then $-\bar{Q}_1$, $-\bar{Q}_2$ and $-\bar{Q}_3$. Let the radii of the conductors be r.

Consider first the potential difference $\bar{V}_{12}$ between conductors 1 and 2, due to the three actual charges on the conductors and the three apparent charges of opposite sign on the images. The difference of potential between conductors 1 and 2 is equal to the algebraic sum of the potential differences which would be pro-

duced by the three charges and the three images each acting separately.

$$\bar{V}_{12} = 2\bar{Q}_1 \log_\epsilon \frac{D}{r} + 2\bar{Q}_2 \log_\epsilon \frac{r}{D} + 2\bar{Q}_3 \log_\epsilon \frac{D}{2D}$$
$$+ 2(-\bar{Q}_1) \log_\epsilon \frac{\sqrt{(2h)^2 + (D)^2}}{2h}$$
$$+ 2(-\bar{Q}_2) \log_\epsilon \frac{2h}{\sqrt{(2h)^2 + (D)^2}}$$
$$+ 2(-\bar{Q}_3) \log_\epsilon \frac{\sqrt{(2h)^2 + (D)^2}}{\sqrt{(2h)^2 + (2D)^2}} \quad (32)$$

Similarly,

$$\bar{V}_{23} = 2\bar{Q}_1 \log_\epsilon \frac{2D}{D} + 2\bar{Q}_2 \log_\epsilon \frac{D}{r} + 2\bar{Q}_3 \log_\epsilon \frac{r}{D}$$
$$+ 2(-\bar{Q}_1) \log_\epsilon \frac{\sqrt{(2h)^2 + (2D)^2}}{\sqrt{(2h)^2 + (D)^2}}$$
$$+ 2(-\bar{Q}_2) \log_\epsilon \frac{\sqrt{(2h)^2 + (D)^2}}{2h}$$
$$+ 2(-\bar{Q}_3) \log_\epsilon \frac{2h}{\sqrt{(2h)^2 + (D)^2}} \quad (33)$$

$$\bar{V}_{31} = 2\bar{Q}_1 \log_\epsilon \frac{r}{2D} + 2\bar{Q}_2 \log_\epsilon \frac{D}{D} + 2\bar{Q}_3 \log_\epsilon \frac{2D}{r}$$
$$+ 2(-\bar{Q}_1) \log_\epsilon \frac{2h}{\sqrt{(2h)^2 + (2D)^2}}$$
$$+ 2(-\bar{Q}_2) \log_\epsilon \frac{\sqrt{(2h)^2 + (D)^2}}{\sqrt{(2h)^2 + (D)^2}}$$
$$+ 2(-\bar{Q}_3) \log_\epsilon \frac{\sqrt{(2h)^2 + (2D)^2}}{2h} \quad (34)$$

The computation can be somewhat simplified by writing equations (32), (33) and (34) in the following forms:

$$\bar{V}_{12} = \bar{Q}_1 \log_\epsilon \left(\frac{D^2}{r^2} \times \frac{4h^2}{4h^2 + D^2}\right) + \bar{Q}_2 \log_\epsilon \left(\frac{r^2}{D^2} \times \frac{4h^2 + D^2}{4h^2}\right)$$
$$+ \bar{Q}_3 \log_\epsilon \left(\frac{1}{4} \times \frac{4h^2 + 4D^2}{4h^2 + D^2}\right) \quad (35)$$

$$\bar{V}_{23} = \bar{Q}_1 \log_\epsilon \left(\frac{4}{1} \times \frac{4h^2 + D^2}{4h^2 + 4D^2}\right) + \bar{Q}_2 \log_\epsilon \left(\frac{D^2}{r^2} \times \frac{4h^2}{4h^2 + D^2}\right)$$
$$+ \bar{Q}_3 \log_\epsilon \left(\frac{r^2}{D^2} \times \frac{4h^2 + D^2}{4h^2}\right) \quad (36)$$

$$\bar{V}_{31} = \bar{Q}_1 \log_e \left(\frac{r^2}{4D^2} \times \frac{h^2 + D^2}{h^2}\right) + \bar{Q}_3 \log_e \left(\frac{4D^2}{r^2} \times \frac{h^2}{h^2 + D^2}\right) \quad (37)$$

Also,

$$\bar{V}_{12} + \bar{V}_{23} + \bar{V}_{31} = 0 \quad (38)$$

$$\bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3 = 0 \quad (39)$$

Knowing $\bar{V}_{12}$, $\bar{V}_{23}$ and $\bar{V}_{31}$ in complex, $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ can be found in complex.

Equations (35), (36) and (37) hold only for electrostatic units. The ratio of the electromagnetic unit of charge to the electrostatic unit of charge is equal to the velocity of light, or 3×10^{10} centimeters per second. The corresponding ratio for the units of voltage is the reciprocal of the velocity of light, or $\frac{1}{3 \times 10^{10}}$ reciprocal centimeters per second. Therefore, if volts are used in equations (35), (36), (37) and (38), the resulting expressions for $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$, derived from these equations, must be multiplied by

$$\left\{10^8 \times \left(\frac{1}{3 \times 10^{10}}\right)\right\} \left\{\left(\frac{1}{3 \times 10^{10}}\right) \times 10\right\} = \frac{1}{9 \times 10^{11}} \quad (40)$$

in order to get the charges in coulombs.

Inspection of equations (35), (36) and (37) shows that the effect of the images on the potentials and hence on the charges is negligible for ordinary heights and spacings of the conductors of transmission lines. For example, consider the first term in equation (35). If the image $-Q_1$ is neglected, this term becomes

$$\bar{Q}_1 \log_e \frac{D^2}{r^2}$$

Suppose D is 10 feet and the height h is 40 feet. Let the radii of the conductors be 0.5 inch. Then the first term actually is

$$2.303\, \bar{Q}_1 \log_{10} \left(\frac{100 \times 144}{0.25} \times \frac{4 \times 1600}{4 \times 1600 + 100}\right) =$$

$$2.303\, \bar{Q}_1 \log_{10} (57{,}600 \times 0.985) = 2.303\, \bar{Q}_1 \times 4.7537$$

If the image is neglected, the first term becomes

$$2.303\, \bar{Q}_1 \log_{10}(57{,}600) = 2.303\, \bar{Q}_1 \times 4.7604$$

Neglecting the image in this case makes an error of only 0.14 of one per cent in the first term of the equation for $\bar{V}_{12}$. Neglect-

ing the images produces errors of the same order of magnitude in the other terms.

If the effect of the images is neglected, equations (35), (36) and (37) become

$$\bar{V}_{12} = \bar{Q}_1 \log_\epsilon \frac{D^2}{r^2} + \bar{Q}_2 \log_\epsilon \frac{r^2}{D^2} + \bar{Q}_3 \log_\epsilon \frac{1}{4} \tag{41}$$

$$\bar{V}_{23} = \bar{Q}_1 \log_\epsilon 4 + \bar{Q}_2 \log_\epsilon \frac{D^2}{r^2} + \bar{Q}_3 \log_\epsilon \frac{r^2}{D^2} \tag{42}$$

$$\bar{V}_{31} = \bar{Q}_1 \log_\epsilon \frac{r^2}{4D^2} + \bar{Q}_3 \log_\epsilon \frac{4D^2}{r^2} \tag{43}$$

The charging current can be obtained from the charges in the following manner:

$$q = Q_m \sin \omega t$$

$$i = \frac{dq}{dt} = \omega Q_m \cos \omega t$$

$$I_{r.m.s} = \frac{1}{\sqrt{2}} \omega Q_m = 2\pi f Q_{r.m.s.} \tag{44}$$

The charging currents in amperes per conductor per centimeter length of line can be found by multiplying the root-mean-square values of the charges per conductor per centimeter length of line by $\omega = 2\pi f$. If the drop in potential along the line is negligible, the total charging current can be found by multiplying the charging current per unit length of line by the length of the line.

Although the solution of the preceding equations for the charges on the conductors of a three-phase line is a simple matter when the conductors are in a plane parallel to the surface of the earth, the equations do not take into account the effect of the poles and steel towers or the effect of adjacent lines. They also neglect the effect of adjacent trees and buildings, irregularities of the earth's surface etc. They assume that the equipotential surface due to the earth is actually coincident with the earth's surface, an assumption which may be considerably in error, especially in very dry sections of the country. This last source of error, however, is of little consequence since the effect of the images on the charges is so small as to be negligible under ordinary conditions. An exact solution taking into account all the factors just mentioned is obviously impossible. In view of these con-

siderations and because of the fact that the charging current is due almost entirely to the capacitance between the conductors of a transmission line when they have the usual height and spacing, it is possible to neglect the effect on the charging current of the earth and adjacent bodies.

Example of the Calculation of the Charging Current of a Three-phase Transmission Line.—A 110,000-volt, three-phase, 60-cycle transmission line has conductors 1.14 inches in diameter, which are in a horizontal plane and are spaced 10 feet between the middle and each outside conductor, at a height of 40 feet above the earth. Assuming there is no drop in voltage along the line, find the charging current in each conductor per mile of line. The line is assumed not to be transposed and the capacitance of the line to earth is neglected. Balanced voltages are assumed. The arrangement of the line and the notation used are the same as in Fig. 137, page 433.

Take $\bar{V}_{12}$ as the axis of reference. Then,

$$\begin{aligned}\bar{V}_{12} &= 110{,}000\,(1 + j0)\\ &= 110{,}000 + j0\\ \bar{V}_{23} &= 110{,}000\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\\ &= -55{,}000 - j95{,}260\\ \bar{V}_{31} &= 110{,}000\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\\ &= -55{,}000 + j95{,}260\end{aligned}$$

From equations (41), (42) and (43), page 436,

$$110{,}000 + j0 = 2.303\left\{\bar{Q}_1 \log_{10}\frac{(10\times 12)^2}{(0.57)^2} + \bar{Q}_2 \log_{10}\frac{(0.57)^2}{(10\times 12)^2} + \bar{Q}_3 \log_{10}\frac{1}{4}\right\} \times 9\times 10^{11} \quad (45)$$

$$-55{,}000 - j95{,}260 = 2.303\left\{\bar{Q}_1 \log_{10} 4 + \bar{Q}_2 \log_{10}\frac{(10\times 12)^2}{(0.57)^2} + \bar{Q}_3 \log_{10}\frac{(0.57)^2}{(10\times 12)^2}\right\} \times 9\times 10^{11} \quad (46)$$

$$-55{,}000 + j95{,}260 = 2.303\left\{\bar{Q}_1 \log_{10}\frac{(0.57)^2}{4\times(10\times 12)^2} + \bar{Q}_3 \log_{10}\frac{4\times(10\times 12)^2}{(0.57)^2}\right\} \times 9\times 10^{11} \quad (47)$$

$$110{,}000 + j0 = 96.31 \times 10^{11}\bar{Q}_1 - 96.31 \times 10^{11}\bar{Q}_2 - 12.48 \times 10^{11}\bar{Q}_3 \quad (48)$$

$$-55{,}000 - j95{,}260 = 12.48 \times 10^{11}\bar{Q}_1 + 96.31 \times 10^{11}\bar{Q}_2 - 96.31 \times 10^{11}\bar{Q}_3 \quad (49)$$

$$-55{,}000 + j95{,}260 = -108.8 \times 10^{11}\bar{Q}_1 + 108.8 \times 10^{11}\bar{Q}_3 \quad (50)$$

$$\bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3 = 0 \quad (51)$$

Substitute in equation (51) the value of $\bar{Q}_2$ from equation (48).

$$\bar{Q}_1 + \left\{\frac{96.31 \times 10^{11}}{96.31 \times 10^{11}}\bar{Q}_1 - \frac{12.48 \times 10^{11}}{96.31 \times 10^{11}}\bar{Q}_3 - \frac{110{,}000}{96.31 \times 10^{11}}\right\} + \bar{Q}_3 = 0$$

$$\bar{Q}_1 + 0.4352\,\bar{Q}_3 - 0.571 \times 10^{-8} = 0 \quad (52)$$

Substitute in equation (50) the value of $\bar{Q}_1$ from equation (52).

$$-55{,}000 + j95{,}260 = -108.8 \times 10^{11}(-0.4352\bar{Q}_3 + 0.571 \times 10^{-8}) + 108.8 \times 10^{11}\bar{Q}_3$$

$$\bar{Q}_3 = 45.63 \times 10^{-11} + j609.8 \times 10^{-11} \text{ coulomb per centimeter} \quad (53)$$

Substitute in equation (50) the value of $\bar{Q}_3$ from equation (53).

$$-55{,}000 + j95{,}260 = -108.8 \times 10^{11}\bar{Q}_1 + 108.8 \times 10^{11}(45.63 \times 10^{-11} + j609.8 \times 10^{-11})$$

$$\bar{Q}_1 = 551 \times 10^{-11} - j265.7 \times 10^{-11} \text{ coulomb per centimeter} \quad (54)$$

Substitute in equation (51) the values of $\bar{Q}_3$ and $\bar{Q}_1$ from equations (53) and (54), respectively.

$$551 \times 10^{-11} - j265.7 \times 10^{-11} + \bar{Q}_2 + 45.63 \times 10^{-11} + j609.8 \times 10^{-11} = 0$$

$$\bar{Q}_2 = -597 \times 10^{-11} - j344.1 \times 10^{-11} \text{ coulomb per centimeter} \quad (55)$$

The numerical values of the charges are

$$Q_1 = \sqrt{(551)^2 + (265.7)^2} \times 10^{-11}$$
$$= 612 \times 10^{-11} \text{ coulomb per centimeter length of line}$$

$$Q_2 = \sqrt{(597)^2 + (344.1)^2} \times 10^{-11}$$
$$= 689 \times 10^{-11} \text{ coulomb per centimeter length of line}$$

$$Q_3 = \sqrt{(45.6)^2 + (609.8)^2} \times 10^{-11}$$
$$= 612 \times 10^{-11} \text{ coulomb per centimeter length of line}$$

To get the charging current in amperes per mile of line, the charges in coulombs per centimeter length of line must be multiplied by

$$
\begin{aligned}
2\pi f(2.540 \times 12 \times 5280) &= 377 \times 160{,}900 \\
&= 6.066 \times 10^7 \\
I_1 &= 612 \times 10^{-11} \times 6.066 \times 10^7 \\
&= 0.371 \text{ ampere per mile of line} \\
I_2 &= 689 \times 10^{-11} \times 6.066 \times 10^7 \\
&= 0.418 \text{ ampere per mile of line} \\
I_3 &= 612 \times 10^{-11} \times 6.066 \times 10^7 \\
&= 0.371 \text{ ampere per mile of line}
\end{aligned}
$$

The currents I_1 and I_3 are equal, since the conductors 1 and 3 are symmetrically placed. If the line is transposed, as all transmission lines are in practice, the average charging current per mile or per unit length of line is the same for all conductors. This assumes balanced voltages.

Capacitance of a Balanced Three-phase Transmission Line with Conductors at the Corners of an Equilateral Triangle, Neglecting the Effect of the Earth.—When the effect of the earth and adjacent bodies is neglected, the expressions for the charging currents and for the capacitance of a balanced three-phase transmission line, with conductors at the corners of an equilateral triangle, become very simple. Refer to Fig. 138.

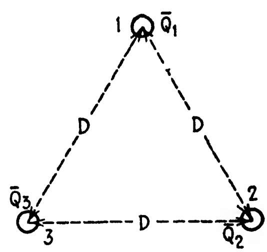


FIG. 138.

Let the distances between the conductors be D, assumed to be large compared with the radii Let the charges on the conductors 1, 2 and 3 be, respectively, $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ electrostatic units per centimeter length of conductor.

Consider the conductors 1 and 2. The charge on conductor 3 cannot produce any difference of potential between conductors 1 and 2, since it is the same distance from each. The potential difference $\bar{V}_{12}$ due to $\bar{Q}_3$ is

$$2\bar{Q}_3 \log_e \frac{D}{D} = 0 \tag{56}$$

The resultant potential difference between conductors 1 and 2, due to all three charges, is that produced by charges $\bar{Q}_1$ and $\bar{Q}_2$ only

$$\begin{aligned}\bar{V}_{12} &= 2\bar{Q}_1 \log_\epsilon \frac{D}{r} + 2\bar{Q}_2 \log_\epsilon \frac{r}{D} \\ &= 2(\bar{Q}_1 - \bar{Q}_2) \log_\epsilon \frac{D}{r} \qquad (57)\end{aligned}$$

in a vector sense, as is indicated in the equation. For balanced condition of impressed voltages, since the line is symmetrical, $\bar{Q}_1$ and $\bar{Q}_2$ must be equal in magnitude and must differ by 120 degrees in time phase. The magnitude of the vector difference of any two equal vectors which differ in time phase by 120 degrees is equal to the magnitude of either vector multiplied by the square root of three. Hence, for balanced conditions, the algebraic form of equation (57) is

$$V = 2\sqrt{3}\, Q \log_\epsilon \frac{D}{r} \qquad (58)$$

or

$$Q = \frac{V}{2\sqrt{3} \log_\epsilon \frac{D}{r}} \qquad (59)$$

Equation (59) gives the magnitude of the charge Q per conductor per centimeter length of line in electrostatic units in terms of the magnitude in electrostatic units of the voltage V between conductors. Since $i = \frac{dq}{dt}$, the magnitude of the charging current per conductor per centimeter length of line is

$$I = \frac{2\pi f V}{2\sqrt{3} \log_\epsilon \frac{D}{r}} \text{ statamperes} \qquad (60)$$

where V is expressed in statvolts, *i.e.*, in electrostatic units. Both I and V are root-mean-square values. If V is expressed in volts, the right-hand member of the equation must be multiplied by $\frac{1}{9 \times 10^{11}}$ to give the current in amperes.

$$I_{amperes} = \frac{2\pi f V_{volts}}{\left(2\sqrt{3}\log_\epsilon \frac{D}{r}\right) 9 \times 10^{11}} \text{ per conductor per centimeter length of line} \quad (61)$$

It is evident from equation (57) that the voltage $\bar{V}_{12}$ is in phase with the vector difference of the charges $\bar{Q}_1$ and $\bar{Q}_2$, *i.e.*, in phase with $(\bar{Q}_1 - \bar{Q}_2)$. Similarly, $\bar{V}_{23}$ is in phase with $(\bar{Q}_2 - \bar{Q}_3)$ and $\bar{V}_{31}$ is in phase with $(\bar{Q}_3 - \bar{Q}_1)$. For balanced conditions, the three charges $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ must be equal in magnitude and must differ by 120 degrees in time phase. The only way in which the charges can be equal in magnitude and differ in phase by 120 degrees, and at the same time have their vector differences taken in pairs in phase with the line voltages, is for the charges to be in phase with the wye voltages of the system.

$$\bar{V}_{12} = \bar{V}_{10} - \bar{V}_{20} = 2\bar{Q}_1 \log_\epsilon \frac{D}{r} - 2\bar{Q}_2 \log_\epsilon \frac{D}{r}$$
$$= (\bar{Q}_1 - \bar{Q}_2)\, 2 \log_\epsilon \frac{D}{r} \quad (62)$$

where the voltages $\bar{V}_{10}$ and $\bar{V}_{20}$ are the voltages between the neutral and conductors 1 and 2, respectively. The phase relations between the voltages and charges are made clearer by referring to Fig. 139. This figure shows the phase relations between the vectors in equation (62).

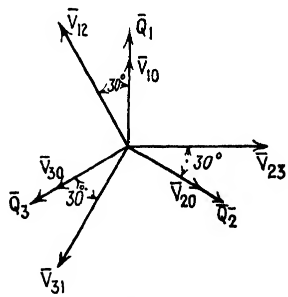


FIG. 139.

The charges on the conductors per unit length of line are the same as would exist on three equal condensers connected in wye across the three-phase line, each condenser having a capacitance of

$$C = \frac{Q}{\frac{V}{\sqrt{3}}} = \frac{Q}{2Q \log_\epsilon \frac{D}{r}} = \frac{1}{2 \log_\epsilon \frac{D}{r}} \text{ statfarads} \quad (63)$$

Equation (63) gives the equivalent capacitance, in electrostatic units, per phase to neutral, per centimeter length of a balanced three-phase transmission line, with conductors at the corners of an equilateral triangle. It gives the capacitance for each of

three equal condensers which, if connected in wye across the line, would take the same charging current as the line actually takes. Equation (63) holds only for balanced conditions.

It should be noticed that the equivalent capacitance to neutral of a balanced three-phase transmission line, with conductors at the corners of an equilateral triangle, is equal to twice the capacitance between conductors of a single-phase line with the same conductor spacing. [Equation (27), page 430.] It is equal to the capacitance to neutral of the single-phase line.

In terms of common logarithms and microfarads per mile of line, equation (63) becomes [see equation (29), page 430]

$$C = \frac{38.8 \times 10^{-3}}{\log_{10} \dfrac{D}{r}} \text{ microfarads per mile of line} \tag{64}$$

The charging current per conductor per mile of a balanced three-phase line with conductors at the corners of an equilateral triangle, neglecting the drop in voltage along the line, is

$$I = \frac{2\pi f V_n \times 38.8 \times 10^{-9}}{\log_{10} \dfrac{D}{r}}$$

$$= \frac{2.44 V_n f \times 10^{-7}}{\log_{10} \dfrac{D}{r}} \text{ ampere per conductor per mile of line} \tag{65}$$

V_n is the voltage to neutral. Equations (57) to (65), inclusive, neglect everything except the line itself. They hold for only balanced conditions.

When the conductors of a balanced transposed transmission line are not at the corners of an equilateral triangle, the average capacitance of the line to neutral may be calculated by using the spacing of the equivalent equilateral arrangement. As an approximation, this spacing may be taken equal to

$$D = \sqrt[3]{D_1 \times D_2 \times D_3} \tag{66}$$

where D_1, D_2 and D_3 are the actual distances between the conductors.

Example of the Calculation of the Average Charging Current per Conductor per Mile of a Transposed Line by the Use of the

Equivalent Equilateral Spacing.—A 110,000-volt, 60-cycle, three-phase, transposed transmission line has its conductors arranged in a horizontal plane 40 feet above the earth, with 10 feet between the middle conductor and each outside conductor. The radius of the conductors is 0.57 inch. What is the average charging current per conductor per mile of line? The equivalent equilateral spacing is used in calculating the charging current.

$$D = \sqrt[3]{D_1 \times D_2 \times D_3}$$
$$= \sqrt[3]{10 \times 10 \times 20}$$
$$= 12.60 \text{ feet}$$
$$I = \frac{2.44 V_n f \times 10^{-7}}{\log_{10} \dfrac{D}{r}}$$

$$I = \frac{2.44 \times \dfrac{110{,}000}{\sqrt{3}} \times 60 \times 10^{-7}}{\log_{10} \dfrac{12.60 \times 12}{0.57}} = \frac{9.30 \times 10^6 \times 10^{-7}}{\log_{10} 265.3}$$
$$= 0.384 \text{ ampere per conductor per mile of line}$$

The actual charging current per conductor per mile of the completely transposed line can be found from the charges per centimeter length of each conductor when the line is not transposed. These charges were found to be (see page 438)

$$\bar{Q}_1 = 551 \times 10^{-11} - j265.7 \times 10^{-11}$$
$$\bar{Q}_2 = -597 \times 10^{-11} - j344.1 \times 10^{-11}$$
$$\bar{Q}_3 = 45.63 \times 10^{-11} + j609.8 \times 10^{-11}$$

coulomb per conductor per centimeter length of line, where $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ are the charges on the three conductors with the conductors in the positions shown in Fig. 137, page 433.

If the line is transposed so that the conductors from left to right, Fig. 137, are 3, 1, 2, the charge $\bar{Q}_1$ on conductor 1 would be the same in magnitude as $\bar{Q}_2$ given above, but it would not be in phase with it, since the middle conductor would now be connected to phase 1 instead of to phase 2. Since the cyclic order of the voltages used in finding the charges $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ was $\bar{V}_{12}$, $\bar{V}_{23}$, $\bar{V}_{31}$ with $\bar{V}_{23}$ lagging $\bar{V}_{12}$, the charge $\bar{Q}_1$ on conductor 1, when it occupies the middle position, *i.e.*, the position 2, Fig. 137, can be found both in magnitude and in time phase by

multiplying $\bar{Q}_2$ by the operator which produces a rotation of +120 degrees.

If $\bar{Q} = \bar{Q}_1$ is the charge on conductor 1 per centimeter length of line when this conductor occupies the position shown in Fig. 137, the charges on it, per centimeter length of line, when it occupies the positions 2 and 3, can be found by rotating the charges $\bar{Q}_2$ and $\bar{Q}_3$ as given above through +120 and −120 degrees, respectively.

The charges on conductor 1 per centimeter length of line when the conductor occupies the three positions necessary for complete transposition are

$$
\begin{aligned}
\text{Position 1: } \bar{Q} &= (551 - j265.7)10^{-11} \\
\text{Position 2: } \bar{Q} &= (-597 - j344.1)(-0.500 + j0.866)10^{-11} \\
&= (596.5 - j344.9)10^{-11} \\
\text{Position 3: } \bar{Q} &= (45.63 + j609.8)(-0.500 - j0.866)10^{-11} \\
&= (505.3 - j344.4)10^{-11} \\
\text{Average: } \bar{Q} &= \frac{1}{3}\{(551 + 596.5 + 505.3) + \\
&\qquad j(-265.7 - 344.9 - 344.4)\}10^{-11} \\
&= (551 - j318)10^{-11} \\
\text{Average: } Q &= \sqrt{(551)^2 + (318)^2} \times 10^{-11} \\
&= 636 \times 10^{-11} \text{ coulomb per centimeter length of line}
\end{aligned}
$$

The actual charging current per conductor per mile of the completely transposed line is

$$
\begin{aligned}
I &= 2\pi f \times 636 \times 10^{-11} \times (2.540 \times 12 \times 5280) \\
&= 0.386 \text{ ampere}
\end{aligned}
$$

This differs by less than one per cent from the value found by the use of the equivalent equilateral spacing.

Voltage Induced in a Telephone or Telegraph Line by the Electrostatic Induction of the Charges on the Conductors of an Adjacent Transmission Line.—A telephone line is on the same poles as a three-phase transmission line, whose conductors are in a plane parallel to the surface of the earth and at a distance h above it. Let the distance between the middle and each of the outside conductors be D. Let the two telephone conductors A and B be at a distance $2d'$ apart and each at a distance h' above

the surface of the earth. Also, let the telephone conductors be at equal distances from a line joining the middle conductor of the transmission line and the image of this conductor.

Refer to Fig. 137, page 433. Let the charges on the conductors of the transmission line per unit length of line be $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ electrostatic units. The potential difference between conductor A of the telephone line and the earth, due to the charges $+\bar{Q}_1$ on conductor 1 of the transmission line and $-\bar{Q}_1$ on the image of conductor 1, is

$$\bar{V}_{AO}' = 2\bar{Q}_1 \log_\epsilon \frac{h}{\sqrt{(h - h')^2 + (D - d')^2}} - 2\bar{Q}_1 \log_\epsilon \frac{h}{\sqrt{(h + h')^2 + (D - d')^2}} \text{ statvolts} \quad (67)$$

Similar expressions hold for the voltages $\bar{V}_{AO}''$ and $\bar{V}_{AO}'''$ between conductor A and the earth, due to the charges on conductors 2 and 3 and their images. The resultant potential difference between conductor A and the earth, due to all three charges $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ and their images $-\bar{Q}_1$, $-\bar{Q}_2$ and $-\bar{Q}_3$, is

$$\bar{V}_{AO} = \bar{Q}_1 \log_\epsilon \frac{(h + h')^2 + (D - d')^2}{(h - h')^2 + (D - d')^2} + \bar{Q}_2 \log_\epsilon \frac{(h + h')^2 + (d')^2}{(h - h')^2 + (d')^2} + \bar{Q}_3 \log_\epsilon \frac{(h + h')^2 + (D + d')^2}{(h - h')^2 + (D + d')^2} \text{ statvolts} \quad (68)$$

If $\bar{V}_{BO}$ is the resultant difference of potential between the conductor B and the earth, found in the same manner as the potential difference $\bar{V}_{AO}$, the potential difference between the conductors A and B of the telephone line is

$$\bar{V}_{AB} = \bar{V}_{AO} - \bar{V}_{BO} \quad (69)$$

In order to evaluate $\bar{V}_{AO}$ and $\bar{V}_{BO}$, it is necessary first to calculate, by the method already given, the charges $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ in complex from the voltage, height and spacing of the conductors of the transmission line. Substitute these values of the charges in equation (68) for $\bar{V}_{AO}$ and in the corresponding equation for $\bar{V}_{BO}$ (not given). The voltage $\bar{V}_{AB}$ then can be found from equation (69). In most cases it is not necessary to consider the effect of the images.

Transposition, when it can be properly carried out for either the power line or the telephone line, eliminates the resultant voltage in an adjacent telephone line due to electromagnetic induction. With the power line transposed, the induced voltages neutralize in such a length of the telephone line as is exposed to a complete transposition of the transmission line. With the telephone line transposed, the voltages induced in adjacent sections of the telephone line are equal and opposite and neutralize. When either the telephone line or the transmission line is transposed, the voltage induced by electromagnetic induction in any section of a telephone line is too low to be troublesome or dangerous.

Transposition, when it is properly carried out for either the telephone line or the transmission line, eliminates the resultant voltage in a telephone line due to electrostatic induction of an adjacent transmission line, *i.e.*, it makes the resultant voltage acting along the line zero. Transposition, however, does not eliminate the danger from the voltage induced by electrostatic induction, as the magnitude of this voltage is independent of the exposed length of the telephone line. The voltage in a telephone line due to the electrostatic induction of an adjacent transmission line, especially a single-phase transmission line with ground return, may not only be troublesome but also dangerous to life. This situation can be met by providing a shunt of low reactance for transmission frequency between the conductors of the telephone line and between these conductors and the earth. This arrangement allows the induced voltages on the telephone line to be dissipated by a flow of current. The shunts, if properly designed, are of sufficiently high reactance for telephonic frequencies to prevent serious leakage of the telephonic current.

When the voltages of the transmission line are unbalanced, the best way to study the effects of electrostatic induction on an adjacent telephone or telegraph line is to resolve the voltages of the transmission line into their positive-phase, negative-phase and zero-phase or residual components. (See Chapter XIII.)

An Example Illustrating the Calculation of the Voltage Induced in a Telephone Line by Electrostatic Induction.—A 110,000-volt, 60-cycle, three-phase transmission line and a telephone line are carried on separated pole lines along the same right-of-way.

The spacings of the conductors of the transmission line, and of the telephone line, the distance between the poles and the height of the conductors above the earth are shown in Fig. 140. The cyclic order of the voltages of the transmission line is 1–2–3 with the voltage 2 lagging the voltage 1. The conductors of the transmission line are No. 0000 B. & S. gauge and have a diameter of 0.46 inch. Find the electrostatic voltage in volts induced on the telephone line. The images are neglected. The voltages of the transmission line are assumed to be balanced.

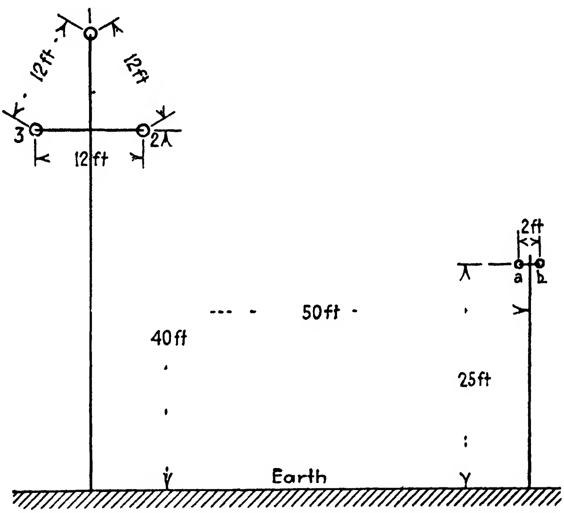


FIG 140

The squares of such distances between conductors as are needed in the solution of this problem are:

$$
\begin{aligned}
(1a)^2 &= (40 - 25 + 12 \times 0.866)^2 + (50 - 1)^2 &&= 3045 \\
(1b)^2 &= (40 - 25 + 12 \times 0.866)^2 + (50 + 1)^2 &&= 3245 \\
(2a)^2 &= (40 - 25)^2 + (50 - 6 - 1)^2 &&= 2074 \\
(2b)^2 &= (40 - 25)^2 + (50 - 6 + 1)^2 &&= 2250 \\
(3a)^2 &= (40 - 25)^2 + (50 + 6 - 1)^2 &&= 3250 \\
(3b)^2 &= (40 - 25)^2 + (50 + 6 + 1)^2 &&= 3474
\end{aligned}
$$

Since the conductors of the transmission line are at the corners of an equilateral triangle, the charge per conductor per centimeter

length of line is [equation (63), page 441, or equation (59), page 440]

$$Q = \frac{V}{\sqrt{3}} \times C = \frac{V}{2\sqrt{3}\log_\epsilon \dfrac{D}{r}}$$

$$= \frac{110{,}000}{2\sqrt{3} \times 2.303 \log_{10} \dfrac{12 \times 12}{0.23}} \times \frac{1}{3} \times 10^{-2}$$

$$= 4930 \times \frac{1}{3} \times 10^{-2}$$

statcoulombs per centimeter length of line.

$$\bar{V}_{ab} = \left\{\bar{Q}_1 \log_\epsilon \left(\frac{1b}{1a}\right)^2 + \bar{Q}_2 \log_\epsilon \left(\frac{2b}{2a}\right)^2 + \bar{Q}_3 \log_\epsilon \left(\frac{3b}{3a}\right)^2\right\} \times 3 \times 10^2$$

volts, where the $\bar{Q}$'s are expressed in statcoulombs.

Taking $\bar{Q}_1$ as an axis of reference,

$$\begin{aligned}
\bar{V}_{ab} &= 2.303 \times 4930\{0.02763(1 + j0) \\
&\qquad + 0.03537(-0.5 - j0.866) \\
&\qquad\qquad + 0.02895(-0.5 + j0.866)\} \text{ volts} \\
&= 11{,}350(0.02763 + j0 - 0.01769 - j0.03063 \\
&\qquad\qquad -0.01447 + j0.02507) \text{ volts} \\
&= 11{,}350 \times (-0.00453 - j0.00556) \text{ volts} \\
V_{ab} &= 11{,}350\sqrt{(0.00453)^2 + (0.00556)^2} \\
&= 81.3 \text{ volts}
\end{aligned}$$

Another Example Illustrating the Calculation of the Voltage Induced in a Telephone Line by Electrostatic Induction.—In the preceding problem, calculate the voltage between conductor a of the telephone line and earth, caused by the electrostatic induction of the transmission line. In this case the images cannot be neglected.

The squares of the distances between conductor a of the telephone line and each conductor of the transmission line and also between conductor a of the telephone line and each of the images of the conductors of the transmission line are needed.

The squares of these distances are:

$$\begin{aligned}
(1a)^2 &= (40 - 25 + 12 \times 0.866)^2 + (50 - 1)^2 = 3045 \\
(2a)^2 &= (40 - 25)^2 + (50 - 6 - 1)^2 = 2074
\end{aligned}$$

$$
\begin{aligned}
(3a)^2 &= (40 - 25)^2 + (50 + 6 - 1)^2 &&= 3250 \\
(1'a)^2 &= (40 + 25 + 12 \times 0.866)^2 + (50 - 1)^2 &&= 8085 \\
(2'a)^2 &= (40 + 25)^2 + (50 - 6 - 1)^2 &&= 6074 \\
(3'a)^2 &= (40 + 25)^2 + (50 + 6 - 1)^2 &&= 7250 \\
(1e)^2 &= (1'e)^2 = (40 + 12 \times 0.866)^2 &&= 2539 \\
(2e)^2 &= (2'e)^2 = (40)^2 &&= 1600 \\
(3e)^2 &= (3'e)^2 = (40)^2 &&= 1600
\end{aligned}
$$

In the preceding table, 1′, 2′ and 3′ indicate images, and the *e's* indicate earth.

From the preceding problem, $Q = 4930 \times \frac{1}{3} \times 10^{-2}$ statcoulombs per centimeter length of line.

$$
\begin{aligned}
\bar{V}_{ae} = \Big\{ &\bar{Q}_1 \log_e \left(\frac{1e}{1a}\right)^2 + \bar{Q}_2 \log_e \left(\frac{2e}{2a}\right)^2 + \bar{Q}_3 \log_e \left(\frac{3e}{3a}\right)^2 \\
&+ (-\bar{Q}_1) \log_e \left(\frac{1'e}{1'a}\right)^2 + (-\bar{Q}_2) \log_e \left(\frac{2'e}{2'a}\right)^2 \\
&+ (-\bar{Q}_3) \log_e \left(\frac{3'e}{3'a}\right)^2 \Big\} \times 3 \times 10^2 \text{ volts}
\end{aligned}
$$

where the $\bar{Q}$'s are expressed in statcoulombs per centimeter length of line.

$$
\bar{V}_{ae} = \left\{ \bar{Q}_1 \log_e \frac{8085}{3045} + \bar{Q}_2 \log_e \frac{6074}{2074} + \bar{Q}_3 \log_e \frac{7250}{3250} \right\} \times 3 \times 10^2 \text{ volts}
$$

Taking $\bar{Q}_1$ as an axis of reference,

$$
\begin{aligned}
\bar{V}_{ae} &= 2.303 \times 4930 \{0.4241 (1 + j0) + 0.4667 (- 0.5 - j0.866) \\
&\qquad 0.3485 (- 0.5 + j0.866)\} \text{ volts} \\
&= 11{,}350 \times (0.0165 - j0.1024) \text{ volts} \\
V_{ae} &= 11{,}350\sqrt{(0.0165)^2 + (0.1024)^2} \\
&= 1173 \text{ volts}
\end{aligned}
$$

CHAPTER XVI

SERIES-PARALLEL CIRCUITS CONTAINING UNIFORMLY DISTRIBUTED RESISTANCE, REACTANCE, CONDUCTANCE AND SUSCEPTANCE

Series-parallel Circuits.—Series-parallel circuits are discussed in Chapter VII. In that chapter, the circuits considered are those whose series impedance and parallel admittance are concentrated between definite points. The circuits are *lumpy* circuits, not *smooth* circuits with uniformly distributed constants.

When the series impedance and parallel admittance of a circuit are uniformly distributed, the equations for the resultant current and voltage, either at the terminals or at any point along the line, are, for steady conditions, comparatively simple if expressed in terms of hyperbolic functions of complex quantities. Tables and charts are now available by means of which such functions may be evaluated.[1] Even if the tables are not at hand, the equations may be solved without difficulty by converting the hyperbolic functions of complex quantities into hyperbolic functions of real quantities, each multiplied by an operator of the form $(\cos \theta + j \sin \theta)$.

The most important example of a circuit having uniformly distributed constants is an ideal transmission line. An ideal transmission line is one which has its conductor resistance and conductor reactance and also its conductance and susceptance between conductors uniformly distributed along the line. The conductor resistance is the ohmic resistance of the conductor, corrected for skin effect. The conductor reactance is discussed in Chapter XIV. The conductance and susceptance between conductors are due to the leakance and capacitance between conductors. The capacitance of a transmission line is discussed in Chapter XV. The leakance between conductors includes the leakage over insulators and poles and the loss caused by dis-

[1] Tables of Complex Hyperbolic and Circular Functions, A. E. Kennelly.

charge through the air due to corona. Under normal operating conditions, the leakage over insulators and poles and the corona discharge should be practically negligible for a properly designed transmission line.

An actual transmission line, when considered between points at which branches are taken off, conforms nearly enough to the conditions for a uniform line to be treated as such without sensible error, *i.e.*, to be treated as a line whose conductor resistance and reactance per unit length of line and whose conductance and susceptance between conductors, also per unit length of line, are constant.

The derivation of the general equations for voltage and current for a uniform line under steady conditions of operation is an easy matter. It involves merely the solution of two simple simultaneous differential equations for voltage and current and the determination of the constants of integration from the terminal conditions of the line.

Equations for Voltage and Current of a Transmission Line Whose Series Resistance and Reactance per Unit Length of Line and Whose Parallel Conductance and Susceptance per Unit Length of Line Are Constant.—The equations which will be developed hold only for steady conditions.

Consider a circuit having uniformly distributed resistance, reactance, conductance and susceptance. Let

r = resistance per conductor per unit length of line
x = reactance per conductor per unit length of line
g = leakage conductance between conductors per unit length of line
b = susceptance between conductors per unit length of line, due to the capacitance between conductors

No actual transmission line can be exactly uniform on account of the variation produced in its constants by many factors, such as temperature, atmospheric pressure, weather conditions etc. Irregularities in the spacing of the conductors affect the reactance, conductance and susceptance of a line. The conductance and susceptance are also affected to some extent by the irregularities in the height of the conductors above the earth's surface. The effect of this variation in height is small for an ordinary

transmission line. The weather conditions affect the corona loss and the leakage loss between conductors. The weather conditions, therefore, affect the conductance between conductors. If a section of a transmission line crosses a high mountain pass, the conductance of the section may be much increased, especially during storms, by the low atmospheric pressure. In practice, average values of the constants are used for calculating the performance of transmission lines.

The resistance r may be determined from the size, material and average temperature of the conductors, due allowance being made for skin effect. The method of calculating the reactance x is given in Chapter XIV. The conductance g may be found by

$$g = \frac{\text{corona loss} + \text{leakage loss (both per unit length of line)}}{(\text{voltage between conductors})^2}$$

The corona loss may be calculated, with sufficient accuracy, by means of empirical formulas.[1] The leakage loss may be determined from existing experimental data. For a well-designed transmission line under good operating conditions, both the corona loss and the leakage loss should be negligible.

The susceptance b may be found from the relation $b = \omega C$, where C is the capacitance between conductors per unit length of line. Capacitance of transmission lines is discussed in Chapter XV.

All values of r, x, g, b, z and y must be given per unit length of line. Although the unit of length chosen is merely a matter of convenience, the mile is commonly used in practice.

$$\bar{z} = r + jx \tag{1}$$

$$\bar{y} = g - jb \tag{2}$$

It must be remembered that b represents capacitive susceptance and is therefore negative. When the value of b is substituted in equation (2), the j term becomes positive because of the two negative signs, one with the jb in the equation, the other with the value of b substituted.

Let p be a point distant L from one end of the line. The fundamental differential equations for the current and voltage at the point p are

[1] Dielectric Phenomena in High-Voltage Engineering, F. W Peek, Jr.

$$d\bar{I} = \bar{y}\bar{V}dL \tag{3}$$
$$d\bar{V} = \bar{z}\bar{I}\,dL \tag{4}$$

from which

$$\frac{d\bar{I}}{dL} = \bar{y}\bar{V} \tag{5}$$
$$\frac{d\bar{V}}{dL} = \bar{z}\bar{I} \tag{6}$$

Equations (5) and (6) are the current and the potential gradients, respectively, at a distance L from one end of the line.

Differentiating equations (5) and (6) with respect to L gives

$$\frac{d^2\bar{I}}{dL^2} = \bar{y}\frac{d\bar{V}}{dL} \tag{7}$$
$$\frac{d^2\bar{V}}{dL^2} = \bar{z}\frac{d\bar{I}}{dL} \tag{8}$$

Combining equations (7) and (6) and equations (8) and (5) gives

$$\frac{d^2\bar{I}}{dL^2} - \bar{y}\bar{z}\bar{I} = 0 \tag{9}$$
$$\frac{d^2\bar{V}}{dL^2} - \bar{y}\bar{z}\bar{V} = 0 \tag{10}$$

Equations (9) and (10) are the differential equations for current and voltage of a transmission line.

Equations (9) and (10) are linear differential equations of the form

$$a_1\frac{d^2p}{dq^2} + a_2\frac{dp}{dq} + a_3p = 0 \tag{11}$$

$p = \epsilon^{mq}$ is a solution of equation (11). The complete solution of equation (11) is

$$p = A_1\epsilon^{m_1q} + A_2\epsilon^{m_2q}$$
$$a_1m^2\epsilon^{mq} + a_2m\epsilon^{mq} + a_3\epsilon^{mq} = 0$$
$$a_1m^2 + a_2m + a_3 = 0$$

From equations (9) and (10), $a_1 = 1$, $a_2 = 0$ and $a_3 = -\bar{y}\bar{z}$.

$$m^2 - \bar{y}\bar{z} = 0$$
$$m = \pm\sqrt{\bar{y}\bar{z}}$$

The solutions of equations (9) and (10) are, therefore,

$$\bar{I} = A_1\epsilon^{L\sqrt{\bar{y}\bar{z}}} + A_2\epsilon^{-L\sqrt{\bar{y}\bar{z}}} \tag{12}$$

$$\bar{V} = B_1\epsilon^{L\sqrt{\bar{y}\bar{z}}} + B_2\epsilon^{-L\sqrt{\bar{y}\bar{z}}} \tag{13}$$

Equations (12) and (13) apply only to the steady state, *i.e.*, after any transient has disappeared.

The constants of integration, A_1, A_2, B_1 and B_2, must be determined from assumed terminal conditions either at the load or at the source of power. Let them be determined from the assumed terminal conditions at the load. For this case, L is the distance from the receiving end of the line to the point at which the voltage and current are desired. It must be reckoned positive when measured from the load end of the line to the source of power. Let the assumed current and voltage at the receiving end or load end of the line, *i.e.*, for $L = 0$, be $\bar{I}_R$ and $\bar{V}_R$, respectively.

Differentiate equation (12) with respect to L and put $L = 0$ and $\bar{V} = \bar{V}_R$. This gives

$$\frac{d\bar{I}}{dL} = A_1\sqrt{\bar{y}\bar{z}}\,\epsilon^{L\sqrt{\bar{y}\bar{z}}} - A_2\sqrt{\bar{y}\bar{z}}\,\epsilon^{-L\sqrt{\bar{y}\bar{z}}} = \bar{y}\bar{V}$$

$$\bar{y}\bar{V}_R = A_1\sqrt{\bar{y}\bar{z}} - A_2\sqrt{\bar{y}\bar{z}} \tag{14}$$

Putting $L = 0$ and $\bar{V} = \bar{V}_R$ directly in equation (12),

$$\bar{I}_R = A_1 + A_2 \tag{15}$$

Combining equations (14) and (15),

$$\bar{y}\bar{V}_R = (\bar{I}_R - A_2)\sqrt{\bar{y}\bar{z}} - A_2\sqrt{\bar{y}\bar{z}}$$

from which

$$A_2 = \frac{\bar{I}_R\sqrt{\bar{y}\bar{z}} - \bar{y}\bar{V}_R}{2\sqrt{\bar{y}\bar{z}}}$$

$$= \frac{1}{2}\left(\bar{I}_R - \sqrt{\frac{\bar{y}}{\bar{z}}}\bar{V}_R\right) \tag{16}$$

and

$$A_1 = \frac{1}{2}\left(\bar{I}_R + \sqrt{\frac{\bar{y}}{\bar{z}}}\bar{V}_R\right) \tag{17}$$

The constants B_1 and B_2 may be found in a similar manner from equation (13):

$$B_2 = \frac{1}{2}\left(\bar{V}_R - \sqrt{\frac{\bar{z}}{\bar{y}}}\bar{I}_R\right) \tag{18}$$

$$B_1 = \frac{1}{2}\left(\bar{V}_R + \sqrt{\frac{\bar{z}}{\bar{y}}}\bar{I}_R\right) \tag{19}$$

Substituting the values of the constants A_1, A_2, B_1 and B_2 in equations (12) and (13) gives the current and voltage at a distance L from the receiving end of the line:

$$\bar{I} = \frac{1}{2}\left(\bar{I}_R + \sqrt{\frac{\bar{y}}{\bar{z}}}\bar{V}_R\right)\epsilon^{L\sqrt{\bar{y}\bar{z}}} + \frac{1}{2}\left(\bar{I}_R - \sqrt{\frac{\bar{y}}{\bar{z}}}\bar{V}_R\right)\epsilon^{-L\sqrt{\bar{y}\bar{z}}} \tag{20}$$

$$= \bar{I}_R\frac{1}{2}\left(\epsilon^{L\sqrt{\bar{y}\bar{z}}} + \epsilon^{-L\sqrt{\bar{y}\bar{z}}}\right) + \bar{V}_R\sqrt{\frac{\bar{y}}{\bar{z}}}\,\frac{1}{2}\left(\epsilon^{L\sqrt{\bar{y}\bar{z}}} - \epsilon^{-L\sqrt{\bar{y}\bar{z}}}\right) \tag{21}$$

$$= \bar{I}_R \cosh L\sqrt{\bar{y}\bar{z}} + \bar{V}_R\sqrt{\frac{\bar{y}}{\bar{z}}} \sinh L\sqrt{\bar{y}\bar{z}} \tag{22}$$

$$\bar{V} = \frac{1}{2}\left(\bar{V}_R + \sqrt{\frac{\bar{z}}{\bar{y}}}\bar{I}_R\right)\epsilon^{L\sqrt{\bar{y}\bar{z}}} + \frac{1}{2}\left(\bar{V}_R - \sqrt{\frac{\bar{z}}{\bar{y}}}\bar{I}_R\right)\epsilon^{-L\sqrt{\bar{y}\bar{z}}} \tag{23}$$

$$= \bar{V}_R\frac{1}{2}\left(\epsilon^{L\sqrt{\bar{y}\bar{z}}} + \epsilon^{-L\sqrt{\bar{y}\bar{z}}}\right) + \bar{I}_R\sqrt{\frac{\bar{z}}{\bar{y}}}\,\frac{1}{2}\left(\epsilon^{L\sqrt{\bar{y}\bar{z}}} - \epsilon^{-L\sqrt{\bar{y}\bar{z}}}\right) \tag{24}$$

$$= \bar{V}_R \cosh L\sqrt{\bar{y}\bar{z}} + \bar{I}_R\sqrt{\frac{\bar{z}}{\bar{y}}} \sinh L\sqrt{\bar{y}\bar{z}} \tag{25}$$

The expressions $\sqrt{\bar{y}\bar{z}}$, $\sqrt{\frac{\bar{y}}{\bar{z}}}$ and $\sqrt{\frac{\bar{z}}{\bar{y}}}$ are the fundamental constants of a transmission line.

If L is measured from the generator end of the line, *i.e.*, is positive from the source of power toward the load, equations (22) and (25) become

$$\bar{I} = \bar{I}_G \cosh L\sqrt{\bar{y}\bar{z}} - \bar{V}_G\sqrt{\frac{\bar{y}}{\bar{z}}} \sinh L\sqrt{\bar{y}\bar{z}} \tag{26}$$

$$\bar{V} = \bar{V}_G \cosh L\sqrt{\bar{y}\bar{z}} - \bar{I}_G\sqrt{\frac{\bar{z}}{\bar{y}}} \sinh L\sqrt{\bar{y}\bar{z}} \tag{27}$$

where $\bar{I}$ and $\bar{V}$ are the current and voltage at a distance L from the generator end of the line. $\bar{I}_G$ and $\bar{V}_G$ are the current and voltage at the generator end of the line.

$\sqrt{\frac{\bar{z}}{\bar{y}}}$ is called the *surge impedance* and is denoted by $\bar{z}_0$. $\sqrt{\frac{\bar{y}}{\bar{z}}}$ is called the *surge admittance* and is denoted by $\bar{y}_0$. Both

the surge impedance and the surge admittance are complex quantities.

$$\bar{z}_0 = \sqrt{\frac{\bar{z}}{\bar{y}}} \qquad \bar{y}_0 = \sqrt{\frac{\bar{y}}{\bar{z}}}$$

$\sqrt{\bar{y}\bar{z}}$ is known as the *propagation constant* and is denoted by $\bar{\alpha}$.

$$\bar{\alpha} = \sqrt{\bar{y}\bar{z}}$$

It follows from Chapter I, pages 22 and 23, that

$$\sqrt{\bar{y}\bar{z}} = \sqrt{yz}\,\underline{\left|\frac{1}{2}(\theta_z + \theta_y)\right.} \tag{28}$$

$$= \sqrt{yz}\,\left\{\cos\frac{1}{2}(\theta_z + \theta_y) + j\sin\frac{1}{2}(\theta_z + \theta_y)\right\} \tag{29}$$

$$= \alpha\underline{|\theta_\alpha}$$

$$= \alpha_1 + j\alpha_2 \tag{30}$$

where

$$\theta_y = \tan^{-1}\frac{b}{g}$$

$$\theta_z = \tan^{-1}\frac{x}{r}$$

$$\theta_\alpha = \frac{1}{2}(\theta_y + \theta_z)$$

$$\alpha_1 = \sqrt{yz}\cos\frac{1}{2}(\theta_y + \theta_z) \tag{31}$$

$$\alpha_2 = \sqrt{yz}\sin\frac{1}{2}(\theta_y + \theta_z) \tag{32}$$

Also,

$$\bar{z}_0 = \sqrt{\frac{\bar{z}}{\bar{y}}} = \sqrt{\frac{z}{y}}\,\underline{\left|\frac{1}{2}(\theta_z - \theta_y)\right.} \tag{33}$$

$$= z_0\underline{|\theta_{z_0}} \tag{34}$$

$$\bar{y}_0 = \sqrt{\frac{\bar{y}}{\bar{z}}} = \sqrt{\frac{y}{z}}\,\underline{\left|\frac{1}{2}(\theta_y - \theta_z)\right.} \tag{35}$$

$$= y_0\underline{|\theta_{y_0}} \tag{36}$$

If $\bar{V}_R$ is taken as the axis of reference and θ_R is the phase angle of the load current $\bar{I}_R$ with respect to the voltage $\bar{V}_R$ at the load, equations (22) and (25) may be written in the following form:

$$\bar{I} = I_R \underline{|\theta_R} \cosh (L\alpha\underline{|\theta_\alpha}) + V_R\underline{|0}\; y_0\underline{|\theta_{y_0}} \sinh (L\alpha\underline{|\theta_\alpha}) \tag{37}$$

$$\bar{V} = V_R\underline{|0} \cosh (L\alpha\underline{|\theta_\alpha}) + I_R \underline{|\theta_R}\; z_0\underline{|\theta_{z_0}} \sinh (L\alpha\underline{|\theta_\alpha}) \tag{38}$$

These equations can be evaluated by the use of tables of hyperbolic functions of complex quantities. (See footnote, page 450.)

Interpretation of Equations (20) and (23).—Substituting $\alpha_1 + j\alpha_2$ for $\sqrt{\bar{y}\bar{z}}$ and $\bar{y}_0$ for $\sqrt{\frac{\bar{y}}{\bar{x}}}$ in equation (20) and remembering that $\epsilon^{L(\alpha_1+j\alpha_2)}$ may be written $\epsilon^{L\alpha_1}\,\epsilon^{jL\alpha_2}$,

$$\bar{I} = \frac{1}{2}(\bar{I}_R + \bar{y}_0\bar{V}_R)\epsilon^{L\alpha_1}\,\epsilon^{jL\alpha_2} + \frac{1}{2}(\bar{I}_R - \bar{y}_0\bar{V}_R)\epsilon^{-L\alpha_1}\,\epsilon^{-jL\alpha_2} \tag{39}$$

For increasing values of L, *i.e.*, in going from the load to the generator, the first term of the second member of equation (39) is a vector which increases in magnitude logarithmically with L on account of the factor $\epsilon^{L\alpha_1}$. This first term advances in phase at a rate which is directly proportional to L, due to the factor $\epsilon^{jL\alpha_2}$. This term may be written

$$\bar{I}_{Direct} = \frac{1}{2}(\bar{I}_R + \bar{y}_0\bar{V}_R)\;\epsilon^{L\alpha_1}(\cos L\alpha_2 + j\sin L\alpha_2)$$

$$= \frac{1}{2}\bar{V}_R(\bar{y}_R + \bar{y}_0)\;\epsilon^{L\alpha_1}\underline{|L\alpha_2} \tag{40}$$

where $\bar{y}_R$ is the admittance of the load. If equation (40) is a vector which increases in length and advances in phase in going from load to generator, it may equally well be represented by a vector which decreases in length and retards in phase in going from generator to load. Hence, the first term of the second member of equation (39) represents a wave traveling from the generator to the load.

The second term of the second member of equation (39) is a vector which decreases in magnitude logarithmically with L, due to the term $\epsilon^{L\alpha_1}$. This term retards in phase at a rate which is directly proportional to L, due to the term $\epsilon^{-jL\alpha_2}$. This term may be written

$$\bar{I}_{Reflected} = \frac{1}{2}(\bar{I}_R - \bar{y}_0\bar{V}_R)\;\epsilon^{-L\alpha_1}(\cos L\alpha_2 - j\sin L\alpha_2)$$

$$= \frac{1}{2}\bar{V}_R(\bar{y}_R - \bar{y}_0)\;\epsilon^{-L\alpha_1}\underline{|-L\alpha_2} \tag{41}$$

$\bar{I}_{Direct}$ is a *direct* wave traveling from the generator to the load. $\bar{I}_{Reflected}$ is a wave *reflected* at the load which travels from the load to the generator.

The two terms of the second member of equation (23) may be considered to represent *direct* and *reflected* voltage waves.

$$\bar{V}_{Direct} = \frac{1}{2}(\bar{V}_R + \bar{z}_0\bar{I}_R)\,\epsilon^{L\alpha_1}\underline{|L\alpha_2}$$

$$= \frac{1}{2}\bar{I}_R(\bar{z}_R + \bar{z}_0)\,\epsilon^{L\alpha_1}\underline{|L\alpha_2} \qquad (42)$$

$$\bar{V}_{Reflected} = \frac{1}{2}(\bar{V}_R - \bar{z}_0\bar{I}_R)\,\epsilon^{-L\alpha_1}\underline{|-L\alpha_2}$$

$$= \frac{1}{2}\bar{I}_R(\bar{z}_R - \bar{z}_0)\,\epsilon^{-L\alpha_1}\underline{|-L\alpha_2} \qquad (43)$$

where $\bar{z}_R$ is the impedance of the load and $\bar{I}_R$ is the load current.

Equations (41) and (43) show that reflection of the current and voltage waves does not occur at the load when $\bar{y}_R = \bar{y}_0$ or $\bar{z}_R = \bar{z}_0$, the equality being considered in a vector sense. In general, reflection always occurs at a point in a circuit where there is a change in the electrical constants of the circuit.

The direct and reflected waves combine to produce standing waves with maximum and minimum points, *i.e.*, loops and nodes, equally spaced along the line. As these loops and nodes must occur a quarter wave length apart, they do not show on lines of present commercial length and frequency, since a quarter wave length, even for 60 cycles, is about 775 miles.

A more complete discussion of direct and reflected waves can be found in works dealing specifically with long-distance power transmission.

Propagation Constant, Attenuation Constant, Wave-length Constant, Velocity of Propagation and Length of Line in Terms of Wave Length.[1]—It was stated on page 456 that

$$\sqrt{\bar{y}\bar{z}} = \alpha_1 + j\alpha_2$$

is known as the *propagation constant.* The real part of the propagation constant, *i.e.*,

$$\alpha_1 = \sqrt{yz}\cos\frac{1}{2}(\theta_y + \theta_z) \qquad (44)$$

[1] The symbols used in this chapter for the propagation and attenuation constants differ from those used for filter circuits in Chap. VIII. (See p. 272.)

determines the attenuation of the traveling waves. It is known as the *attenuation constant.* [Equation (29), page 456.]

The coefficient of the imaginary part of the propagation constant, *i.e.*,

$$\alpha_2 = \sqrt{yz} \sin \frac{1}{2} (\theta_y + \theta_z) \tag{45}$$

determines the amount of phase retardation of the component waves in their direction of propagation. It is known as the *wave-length constant.* [Equation (29), page 456.]

The wave length λ of the traveling waves is

$$\lambda = \frac{2\pi}{\alpha_2} \tag{46}$$

where α_2 is in radians. The unit in which the wave length λ is expressed depends on the unit of length selected for the constants y and z. If they are per mile of line, λ is in miles. If they are in any other unit, λ is in the same unit. The magnitude of the product yz is independent of the units chosen for admittance and impedance, provided corresponding units are used, *i.e.*, mhos and ohms or abmhos and abohms. Substituting α_2 from equation (32), page 456, in equation (46) gives

$$\lambda = \frac{2\pi}{\sqrt{yz} \sin \frac{1}{2} (\theta_y + \theta_z)} \tag{47}$$

The length L of a transmission line, expressed in terms of the wave length for the impressed frequency, is

$$\frac{L}{\lambda} = \frac{\alpha_2 L}{2\pi} = \frac{L\sqrt{yz} \sin \frac{1}{2} (\theta_y + \theta_z)}{2\pi} \tag{48}$$

The velocity of propagation is

$$\lambda f = \frac{2\pi f}{\alpha_2} \tag{49}$$

where f is the frequency.

$$\begin{aligned} \bar{y}\bar{z} &= (g - jb)(r + jx) = (\alpha_1 + j\alpha_2)^2 \\ (rg + xb) + j(-rb + gx) &= \alpha_1^2 + j2\alpha_1\alpha_2 - \alpha_2^2 \\ \alpha_1^2 - \alpha_2^2 &= rg + xb \\ 2\alpha_1\alpha_2 &= gx - rb \end{aligned}$$

Solving for α_1 and α_2,

$$\alpha_1 = \sqrt{\frac{1}{2}\,(zy + xb + rg)} \tag{50}$$

$$\alpha_2 = \sqrt{\frac{1}{2}\,(zy - xb - rg)} \tag{51}$$

If there were no transmission losses, r and g would each be zero. Under this condition, there would be no attenuation and the wave-length constant would be

$$\begin{aligned}\alpha_2 &= \sqrt{(x)(-b)}\\ &= \omega\sqrt{LC}\end{aligned} \tag{52}$$

where $\omega = 2\pi f$ and L and C are the inductance and capacitance per unit length of line. It must be remembered that b in equation (52) is capacitive susceptance and is therefore negative.

For a line without transmission losses, the wave length would be

$$\lambda = \frac{2\pi}{\alpha_2} = \frac{2\pi}{\omega\sqrt{LC}} = \frac{1}{f\sqrt{LC}} \tag{53}$$

From the dimensions of inductance and capacitance, it can be shown that $\dfrac{1}{\sqrt{LC}}$ has the dimensions of velocity. This velocity has been shown to be the velocity of light, or 3×10^{10} centimeters per second. It is the limiting velocity of propagation for electric waves. The actual velocity of propagation is slightly less than this, on account of the terms zy and $-rg$ in the expression for α_2 [Equation (51)]. The difference between the actual velocity and that found by assuming no transmission losses is not great, as a rule.

Solution of the Transmission Line.—If a table of hyperbolic functions of complex quantities is available, equations (22) and (25) or (37) and (38), pages 455 and 457, should be used for calculating $\bar{I}_G$ and $\bar{V}_G$ at the generator when $\bar{I}_R$ and $\bar{V}_R$ at the load are known. For calculating $\bar{I}_R$ and $\bar{V}_R$ at the load when $\bar{I}_G$ and $\bar{V}_G$ at the generator are known, equations (26) and (27), page 455, should be used.

The open-circuit voltage at the load is found by putting $\bar{I}_R$ equal to zero in equation (25), page 455, or in equation (38), page 457, and then solving for $\bar{V}_R$.

The charging current of the line, *i.e.*, the no-load current, may be found for fixed generator voltage by substituting, in equation (22), page 455, or in equation (37), page 457, the no-load voltage at the receiving or load end of the line, as found above, and then solving for $\bar{I}_G$, $\bar{I}_R$ being zero.

The length of the line in terms of the wave length of the impressed voltage is found from equation (48), page 459.

The efficiency of transmission is

$$\frac{P_R}{P_G} = \frac{V_R I_R \cos\theta_R}{V_G I_G \cos\theta_G}$$

where P_R and P_G are the power at the load end and the generator end of the line, respectively. The angles θ_R and θ_G are the power-factor angles at the load end and the generator end of the line, respectively. The power may be found as indicated, but it may be more convenient to find it from the complex expressions for current and voltage.

Evaluation of the Equations for Current and Voltage when Tables of Hyperbolic Functions of Complex Quantities Are not Available.—From equation (30), page 456,

$$\begin{aligned}
\cosh L\bar{\alpha} &= \cosh L(\alpha_1 + j\alpha_2) \\
&= \frac{1}{2}\{\epsilon^{(L\alpha_1 + jL\alpha_2)} + \epsilon^{-(L\alpha_1 + jL\alpha_2)}\} \\
&= \frac{1}{2}\,\epsilon^{L\alpha_1}(\cos L\alpha_2 + j\sin L\alpha_2) \\
&\qquad + \frac{1}{2}\,\epsilon^{-L\alpha_1}(\cos L\alpha_2 - j\sin L\alpha_2) \\
&= \frac{1}{2}(\epsilon^{L\alpha_1} + \epsilon^{-L\alpha_1})\cos L\alpha_2 \\
&\qquad + j\frac{1}{2}(\epsilon^{L\alpha_1} - \epsilon^{-L\alpha_1})\sin L\alpha_2 \\
&= \cosh L\alpha_1 \cos L\alpha_2 + j\sinh L\alpha_1 \sin L\alpha_2 \qquad (54)
\end{aligned}$$

It may be shown, in a similar manner, that

$$\sinh L\bar{\alpha} = \sinh L\alpha_1 \cos L\alpha_2 + j\cosh L\alpha_1 \sin L\alpha_2 \qquad (55)$$

Then,

$$\begin{aligned}
\bar{V}_G &= \bar{V}_R \cosh L\bar{\alpha} + \bar{I}_R \bar{z}_0 \sinh L\bar{\alpha} \\
&= \bar{V}_R(\cosh L\alpha_1 \cos L\alpha_2 + j\sinh L\alpha_1 \sin L\alpha_2) \\
&\qquad + \bar{I}_R \bar{z}_0(\sinh L\alpha_1 \cos L\alpha_2 + j\cosh L\alpha_1 \sin L\alpha_2) \qquad (56)
\end{aligned}$$

where

$$\bar{z}_0 = z_0\{\cos\frac{1}{2}(\theta_z - \theta_y) + j\sin\frac{1}{2}(\theta_z - \theta_y)\}$$

$$\begin{aligned}\bar{I}_G &= \bar{I}_R\cosh L\bar{\alpha} + \bar{V}_R\bar{y}_0\sinh L\bar{\alpha}\\ &= \bar{I}_R(\cosh L\alpha_1\cos L\alpha_2 + j\sinh L\alpha_1\sin L\alpha_2)\\ &\quad + \bar{V}_R\bar{y}_0(\sinh L\alpha_1\cos L\alpha_2 + j\cosh L\alpha_1\sin L\alpha_2)\end{aligned} \tag{57}$$

where

$$\bar{y}_0 = y_0\{\cos\frac{1}{2}(\theta_y - \theta_z) + j\sin\frac{1}{2}(\theta_y - \theta_z)\}$$

$$\alpha_1 = \sqrt{yz}\cos\frac{1}{2}(\theta_y + \theta_z)$$

$$\alpha_2 = \sqrt{yz}\sin\frac{1}{2}(\theta_y + \theta_z)$$

$$z_0 = \sqrt{\frac{z}{y}} \qquad y_0 = \sqrt{\frac{y}{z}}$$

$\bar{V}_G$, $\bar{V}_R$, $\bar{I}_G$ and $\bar{I}_R$ must be in complex. When $\bar{V}_R$ and $\bar{I}_R$ are known in complex, equations (56) and (57) give $\bar{V}_G$ and $\bar{I}_G$ in complex. When $\bar{V}_G$ and $\bar{I}_G$ are known in complex, these equations give $\bar{V}_R$ and $\bar{I}_R$ in complex.

The open-circuit voltage at the load or receiving end of the line is found by putting $\bar{I}_R$ equal to zero in equation (56).

$$\bar{V}_R(\text{open circuit}) = \frac{\bar{V}_G}{\cosh L\alpha_1\cos L\alpha_2 + j\sinh L\alpha_1\sin L\alpha_2} \tag{58}$$

When $\bar{I}_R$ is zero, the first term of the second member of equation (57) is zero. If $\bar{V}_R'$ is the open-circuit voltage at the load end of the line, found from equation (58), the charging current is given by

$$\begin{aligned}\bar{I}_G(\text{charging}) &= \bar{V}_R'y_0\{\cos\frac{1}{2}(\theta_y - \theta_z) + j\sin\frac{1}{2}(\theta_y - \theta_z)\}\\ &\quad\times\{\sinh L\alpha_1\cos L\alpha_2 + j\cosh L\alpha_1\sin L\alpha_2\}\end{aligned} \tag{59}$$

When a transmission line is not uniform, due to loads being taken off at intermediate points or to a change in conductor spacing or in size of conductors, it must be considered in sections of such length that the principles of uniform lines are applicable. The ends of the sections are at the points where loads are applied or where the conductor spacing or the size of conductors changes.

Example of the Calculation of the Performance of a Transmission Line for Steady Operating Conditions from Its Electrical Constants.—A 150-mile, 25-cycle, 110,000-volt, three-phase transmission line has its conductors placed at the corners of an equilateral triangle. The conductors are 10 feet apart on centers. Each conductor is a 19-strand copper cable with an overall diameter of $\frac{5}{8}$ inch and an equivalent cross section of 300,000 circular mils. The skin effect at 25 cycles for conductors of the size used for this line is very small and may be neglected. The line operates below its corona voltage, *i.e.*, below the voltage at which the loss due to corona discharge between conductors appears. The line insulation is assumed high. The corona loss and the leakage between conductors over line insulators etc. are assumed to be negligible. Calculations are based on an average line temperature of 20 degrees centigrade.

What are the voltage, current, power and power factor at the power station or generator end of the line, for an inductive load at the receiving end of 30,000 kilowatts at 110,000 volts and 0.90 power factor? What is the efficiency of transmission?

If the voltage at the generator end of the line remains constant at the value required for the given load conditions, what is the charging current of the line and the open-circuit voltage at the receiving or load end of the line?

What is the length of the line in terms of the wave length of the current and voltage waves?

The resistance of the copper conductors is assumed to be 10.58 ohms per mil-foot at 20 degrees centigrade. The mile is used as the unit of length. The radius of the conductors is $\frac{1}{2}(\frac{5}{8}) = 0.3125$ inch.

$$r = 10.58 \times \frac{5280}{300{,}000} = 0.1862 \text{ ohm per mile}$$

$$g = \text{negligible (assumed zero)}$$

$$C \text{ (to neutral)} = \frac{38.8 \times 10^{-3}}{\log_{10} \dfrac{10 \times 12}{0.3125}} = \frac{38.8 \times 10^{-3}}{2.584}$$

$$= 15.0 \times 10^{-3} \text{ microfarad per mile}$$

$$b = -2\pi f \times 15.0 \times 10^{-3} \times 10^{-6}$$

$$= -157 \times 15.0 \times 10^{-9}$$

$$= -2.355 \times 10^{-6} \text{ mho per mile}$$

$$x = 2\pi f\left(741 \log_{10} \frac{10 \times 12}{0.3125} + 80\right) \times 10^{-6}$$
$$= 157(741 \times 2.584 + 80) \times 10^{-6}$$
$$= 0.3130 \text{ ohm per mile}$$

$$z = \sqrt{r^2 + x^2} = \sqrt{(0.1862)^2 + (0.3130)^2}$$
$$= 0.3642 \text{ ohm per mile}$$

$$\tan \theta_z = \frac{0.3130}{0.1862} = 1.681$$
$$\theta_z = +59.25 \text{ degrees per mile}$$
$$\bar{z} = 0.3642\underline{\vert + 59°25} \text{ ohm per mile}$$

$$y = \sqrt{g^2 + b^2} = \sqrt{(0.000)^2 + (2.355 \times 10^{-6})^2}$$
$$= 2.355 \times 10^{-6} \text{ mho per mile}$$

$$\tan \theta_y = \frac{-(-2.355) \times 10^{-6}}{0.000} = \infty$$
$$\theta_y = +90.0 \text{ degrees per mile}$$
$$\bar{y} = 2.355 \times 10^{-6}\underline{\vert +90°0} \text{ ohm per mile}$$

$$\alpha = \sqrt{yz} = \sqrt{2.355 \times 10^{-6} \times 0.3642}$$
$$= 0.9261 \times 10^{-3} \text{ per mile}$$

$$\theta_\alpha = \frac{\theta_y + \theta_z}{2} = \frac{90°0 + 59°25}{2}$$
$$= 74.63 \text{ degrees per mile}$$
$$\bar{\alpha} = 0.9261 \times 10^{-3}\underline{\vert +74°63} \text{ per mile}$$

$$z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{0.3642}{2.355 \times 10^{-6}}}$$
$$= 393.3 \text{ ohms}$$

$$\theta_{z_0} = \frac{\theta_z - \theta_y}{2} = \frac{59°25 - 90°00}{2}$$
$$= -15.38 \text{ degrees}$$
$$\bar{z}_0 = 393.3\underline{\vert - 15°38} \text{ ohms}$$

$$y_0 = \frac{1}{z_0} = \frac{1}{393.3}$$
$$= 0.002543 \text{ mho}$$
$$\theta_{y_0} = -\theta_{z_0} = +15°38 \text{ degrees}$$
$$\bar{y}_0 = 0.002543\underline{\vert +15°38} \text{ mho}$$

$$\sin \theta_\alpha = 0.9643$$
$$\cos \theta_\alpha = 0.2651$$
$$\sin \theta_{z_0} = -0.2651$$

$$\cos\theta_{x_0} = 0.9643$$
$$\sin\theta_{y_0} = 0.2651$$
$$\cos\theta_{y_0} = 0.9643$$
$$\alpha_1 = \alpha\cos\theta_\alpha = 0.9261 \times 10^{-3} \times 0.2651$$
$$= 0.2455 \times 10^{-3} \text{ hyperbolic radian per mile}$$
$$\alpha_2 = \alpha\sin\theta_\alpha = 0.9261 \times 10^{-3} \times 0.9643$$
$$= 0.8931 \times 10^{-3} \text{ radian per mile}$$
$$\alpha_1 L = 0.2455 \times 10^{-3} \times 150$$
$$= 0.03683 \text{ hyperbolic radian}$$
$$\alpha_2 L = 0.8931 \times 10^{-3} \times 150$$
$$= 0.1340 \text{ radian}$$
$$V_R \text{ (to neutral)} = \frac{110{,}000}{\sqrt{3}} = 63{,}510 \text{ volts}$$
$$I_R \text{ (per conductor)} \frac{30{,}000 \times 1000}{3 \times 63{,}510 \times 0.90} = 175.0 \text{ amperes}$$
$$\cos\theta_R = 0.90 \qquad \theta_R = 25.84 \text{ degrees}$$

Take $\bar{V}_R$ as the axis of reference.

$$\bar{V}_R = 63{,}510 \underline{/0^\circ.0000}$$
$$\bar{I}_R = 175.0 \underline{/-25^\circ.84}$$
$$\cosh\alpha_1 L = 1.001$$
$$\sinh\alpha_1 L = 0.03684$$
$$\alpha_2 L = 0.1340 \times \frac{180^\circ}{\pi}$$
$$= 7.675 \text{ degrees}$$
$$\sin\alpha_2 L = 0.1336$$
$$\cos\alpha_2 L = 0.9912$$
$$\cosh\bar{\alpha}L = \cosh\alpha_1 L\cos\alpha_2 L + j\sinh\alpha_1 L\sin\alpha_2 L$$
$$= 1.001 \times 0.9912 + j0.03684 \times 0.1336$$
$$= 0.9922 + j0.004920$$
$$= \sqrt{(0.9922)^2 + (0.004920)^2}\,\underline{/\tan^{-1}\frac{0.004920}{0.9922}}$$
$$= 0.9922 \underline{/+0^\circ.284}$$
$$\sinh\bar{\alpha}L = \sinh\alpha_1 L\cos\alpha_2 L + j\cosh\alpha_1 L\sin\alpha_2 L$$
$$= 0.03684 \times 0.9912 + j\,1.001 \times 0.1336$$
$$= 0.03651 + j\,0.1337$$
$$= \sqrt{(0.03651)^2 + (0.1337)^2}\,\underline{/\tan^{-1}\frac{0.1337}{0.03651}}$$
$$= 0.1386 \underline{/+74^\circ.72}$$

$$\bar{V}_G = \bar{V}_R \cosh \bar{\alpha}L + \bar{I}_R \bar{z}_0 \sinh \bar{\alpha}L$$
$$= 63{,}510\underline{|0°000} \times 0.9922\underline{|+0°28}$$
$$+ 175.0\underline{|-25°84} \times 393.3\underline{|-15°38} \times 0.1386\underline{|+74°74}$$
$$= 63{,}000\underline{|+0°28} + 9538\underline{|+33°50}$$
$$= 63{,}000(1.000 + j0.00495) + 9538(0.8338 + j0.5521)$$
$$= 70{,}950 + j5578$$
$$= \sqrt{(70{,}950)^2 + (5578)^2}\underline{|\tan^{-1} \frac{5578}{70{,}950}}$$
$$= 71{,}170\underline{|+4°495} \text{ volts to neutral}$$
$$V_G = 71{,}170 \times \sqrt{3} = 123{,}300 \text{ volts between conductors}$$

$$\text{Line regulation for the given load} = \frac{123{,}300 - 110{,}000}{110{,}000} \times 100$$
$$= 12.1 \text{ per cent}$$

The no-load voltage at the load with 123,300 volts maintained at the generator is

$$V_R \text{ (no load)} = \frac{V_G}{\sqrt{(\cosh \alpha_1 L \cos \alpha_2 L)^2 + (\sinh \alpha_1 L \sin \alpha_2 L)^2}}$$
$$= \frac{123{,}300}{0.9922}$$
$$= 124{,}300 \text{ volts between conductors}$$

The percentage rise in voltage at the load end when the load is removed, the generator voltage being maintained at 123,300 volts, is

$$\frac{124{,}300 - 110{,}000}{110{,}000} \times 100 = 13.0 \text{ per cent}$$

$$\bar{I}_G = \bar{I}_R \cosh \bar{\alpha}L + \bar{V}_R \bar{y}_0 \sinh \bar{\alpha}L$$
$$= 175.0\underline{|-25°84} \times 0.9922\underline{|+0°284}$$
$$+ 63{,}510\underline{|0°000} \times 0.002543\underline{|+15°38} \times 0.1386\underline{|+74°72}$$
$$= 173.6\underline{|-25°56} + 22.38\underline{|+90°10}$$
$$= 173.6\ (0.9022 - j0.4314) + 22.38(-0.0017 + j1.000)$$
$$= 156.6 - j52.51$$
$$= \sqrt{(156.6)^2 + (52.51)^2}\underline{|\tan^{-1} \frac{-52.51}{156.6}}$$
$$= 165.2\underline{|-18°54} \text{ amperes}$$
$$\theta_G = \theta_V - \theta_I$$
$$= 4°495 - (-18°54)$$
$$= 23.03 \text{ degrees}$$

Power factor at generator $= \cos\theta_G = \cos 23.°03$
$= 0.9205$

$$P_G = V_G I_G \cos\theta_G$$
$$= 71{,}170 \times 165.2 \times 0.9205 \times 10^{-3}$$
$$= 10{,}820 \text{ kilowatts per phase}$$

By complex method,

$$P_G = (70{,}950 \times 156.6 - 5578 \times 52.51) \times 10^{-3}$$
$$= 10{,}820 \text{ kilowatts per phase}$$

Efficiency of transmission $= \dfrac{10{,}000}{10{,}820} \times 100$
$= 92.5$ per cent

When the generator voltage is maintained at the value required to give 110,000 volts between conductors at the load, the load being 30,000 kilowatts at 0.9 power factor (inductive), the no-load charging current is

$$\bar{I}_G(\text{charging}) = \bar{I}_G' = \bar{V}_R'\bar{y}_0 \sinh \bar{\alpha}L$$

where $\bar{V}_R'$ is the voltage at the load on open circuit.

$$\bar{I}_G' = \frac{\bar{V}_R'}{\bar{V}_R}(\bar{V}_R\bar{y}_0 \sinh \bar{\alpha}L)$$

The magnitude of $(\bar{V}_R\bar{y}_0 \sinh \bar{\alpha}L)$ has already been found for a voltage at the load of $V_R = \dfrac{110{,}000}{\sqrt{3}}$ volts to neutral. (See calculation of the generator current for load conditions.) It is the second term of the second member in the expression for the generator current. The charging current at no load varies directly with the voltage.

$$I' = \frac{123{,}300}{110{,}000} \times 22.38$$
$$= 25.1 \text{ amperes}$$

Length of the line in wave lengths is

$$L_\lambda = \frac{\alpha_2 L}{2\pi}\lambda$$
$$= \frac{0.1340}{2 \times 3.142}\lambda = 0.02132\lambda$$

where λ is the wave length of the current and voltage waves.

Length of line, neglecting the transmission losses, is

$$L_\lambda = \frac{\text{length of line in miles} \times \text{frequency}}{\text{velocity of light in miles per second}} \times \text{wave length}$$

$$= \frac{150 \times 25}{186{,}000} \times \lambda$$

$$= 0.02016\lambda$$

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